UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2012/2013

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS EIGHT PAGES, INCLUDING THIS PAGE.

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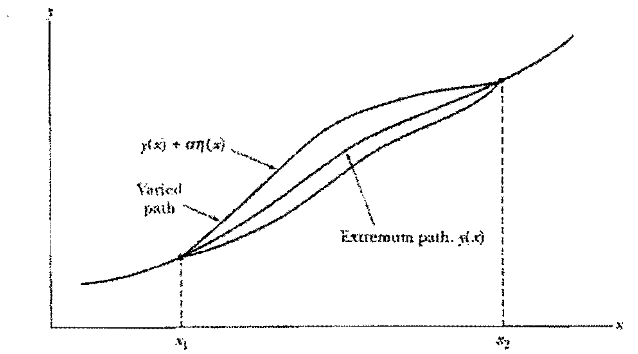
P320 CLASSICAL MECHANICS

Question one

(a) Given the following definite integral $J(\alpha)=\int_{x_1}^{x_2} f(y(\alpha,x),y'(\alpha,x);x)\,dx$,
where the varied integration path is $y(\alpha,x)=y(x)+\alpha\,\eta(x)$ *and* $\eta(x_1)=\eta(x_2)=0$ as shown in the following diagram :



Using the extremum condition for $J(\alpha)$, i.e., $\left.\dfrac{\partial J(\alpha)}{\partial \alpha}\right|_{\alpha=0}=0$, to deduce that f *along the extremum path ,i.e.,* $f(y(x),y'(x);x)$, satisfies the following equation:

$\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right)=0$. **(10 marks)**

(b) If H denotes the Hamiltonian function and L is the Lagrangian function, use the definition $H=\sum_{\alpha=1}^{n} p_\alpha\,\dot{q}_\alpha - L$ (where p_α *and* q_α $(\alpha=1,2,\cdots,n)$ are the generalized momenta and coordinates respectively, i.e., $H=H(q_1,\cdots,q_n,p_1,\cdots,p_n,t)$, $L=L(q_1,\cdots,q_n,\dot{q}_1,\cdots,\dot{q}_n,t)$, $p_\alpha=\dfrac{\partial L}{\partial \dot{q}_\alpha}$ *and* $\dot{p}_\alpha=\dfrac{\partial L}{\partial q_\alpha}$) to show that

(i) $\dot{q}_\alpha=\dfrac{\partial H}{\partial p_\alpha}$ $\alpha=1,2,\cdots,n$ **(4 marks)**

(ii) $\dot{p}_\alpha=-\dfrac{\partial H}{\partial q_\alpha}$ $\alpha=1,2,\cdots,n$ **(4 marks)**

(iii) $\dfrac{\partial H}{\partial t}=-\dfrac{\partial L}{\partial t}$ **(7 marks)**

Question two

(a) The Poisson Bracket $[F,G]$ of two functions F *and* G of canonical variables p_α *and* q_α is given by

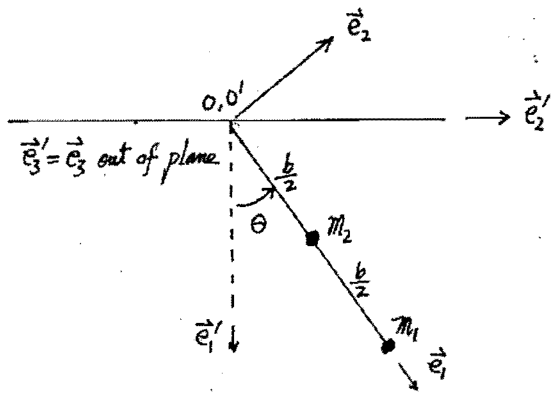
$$[F,G] \equiv \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}-\frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right) .$$

Show that

(i) $[H,q_\alpha]=-\dot{q}_\alpha$ where H is the Hamiltonian, **(4 marks)**

(ii) $[p_\alpha , p_\beta]=0$ where $\begin{cases}\alpha=1,2,\cdots,n\\ \beta=1,2,\cdots,n\end{cases}$, **(3 marks)**

(b) A pendulum is composed of a rigid rod of length b with a mass m_1 at its end. Another mass m_2 is placed halfway down the rod. The mass of the rod itself is negligible. Let the fixed and the body coordinate systems have their origin at the pendulum pivot point. Let $(\vec{e}_1{}' , \vec{e}_2{}' , \vec{e}_3{}')$ *and* $(\vec{e}_1 , \vec{e}_2 , \vec{e}_3)$ be the unit vectors of the fixed and the body coordinate system respectively as shown below.



(i) Write down the inertia tensor I for the pendulum with respect to the body coordinate system given above and deduce that I is a diagonal matrix with its diagonal elements as $I_{1,1}=0$ *and* $I_{2,2}=I_{3,3}=\left(m_1+\frac{m_2}{4}\right)b^2$. **(6 marks)**

(ii) From the equation of rotational motion , i.e., $\dot{\vec{L}}=\vec{N}$ where angular momentum $\vec{L}=I\,\vec{\omega}$ and torque $\vec{N}=\sum_\alpha\left(\vec{r}_\alpha\times\vec{F}_\alpha\right)$, deduce the following equation :

$$b^2\left(m_1+\frac{m_2}{4}\right)\ddot{\theta}=-\,b\,g\sin(\theta)\left(m_1+\frac{m_2}{2}\right)$$ **(12 marks)**

(Hint :

$\vec{\omega}=\vec{e}_3{}'\dot{\theta}$, $\vec{F}_1=\vec{e}_1{}'m_1\,g$, $\vec{F}_2=\vec{e}_1{}'m_2\,g$,

$\vec{r}_1=\vec{e}_1{}'b\cos(\theta)+\vec{e}_2{}'b\sin(\theta)$ *and* $\vec{r}_2=\vec{e}_1{}'\frac{b}{2}\cos(\theta)+\vec{e}_2{}'\frac{b}{2}\sin(\theta)$)

Question three

(a) For circular orbits in an attractive central force potential of the form $V = -\dfrac{k}{r^n}$ where k is a positive constant and $n > 0$, find a relation between the kinetic and potential energies and show that

$$T = \frac{n\,k}{2\,r^n} \quad .$$ **(9 marks)**

(Hint : $\vec{a} = \vec{e}_r \left(\ddot{r} - r\,\dot{\theta}^2\right) + \vec{e}_\theta \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)$)

(b) Starting from the law of conservation of angular momentum l, derive Kepler's third law, i.e., the relation between the period τ of a closed orbit in an attractive inverse square central force and the area A of the orbit. Show that

$$\tau = \frac{2\,\mu}{l}\,A$$ where μ is the reduced mass of the system. **(7 marks)**

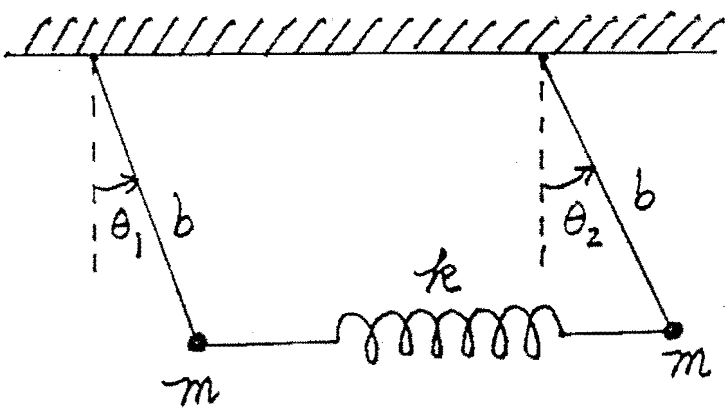
(c) An earth satellite moves in an elliptical orbit with period τ, eccentricity ε and semi-major axis a. The maximum radial velocity, named as $v_{\theta,\max}$, occurs at $r = r_{\min}$. Show that

$$v_{\theta,\max} = \frac{2\,\pi\,a}{\tau\,\sqrt{1-\varepsilon^2}}$$ **(9 marks)**

(Hint : $\mu\, r_{\min}\, v_{\theta,\max} = l$, $A = \pi\, a\, b$ *and* $b = a\sqrt{1-\varepsilon^2}$)

Question four

Two pendulums of equal lengths b and equal masses m are connected by a spring of force constant k as shown below. The spring is unstretched in the equilibrium position, i.e., $\theta_1 = 0 \ and \ \theta_2 = 0$.



(i) For small $\theta_1 \ and \ \theta_2$, i.e.,
$\left(\sin(\theta_1) \approx \theta_1 \ , \ \sin(\theta_2) \approx \theta_2 \ , \ \cos(\theta_1) \approx 1 - \frac{\theta_1^2}{2} \ and \ \cos(\theta_2) \approx 1 - \frac{\theta_2^2}{2} \right)$, show that the Lagrangian for the system can be expressed as:

$$L = \frac{1}{2} m b^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{m g b}{2}\left(\theta_1^2 + \theta_2^2\right) - \frac{k b^2}{2}(\theta_1 - \theta_2)^2$$

where the zero gravitational potential is set at the equilibrium position. **(8 marks)**

(ii) Write down the equations of motion and deduce that

$$\begin{cases} \ddot{\theta}_1 = -\left(\dfrac{m g + k b}{m b}\right)\theta_1 + \dfrac{k}{m}\theta_2 \\ \ddot{\theta}_2 = \dfrac{k}{m}\theta_1 - \left(\dfrac{m g + k b}{m b}\right)\theta_2 \end{cases}$$

(6 marks)

(iii) Set $\theta_1 = \hat{X}_1 \, e^{i\omega t} \ and \ \theta_2 = \hat{X}_2 \, e^{i\omega t}$ (where $\hat{X}_1 \ and \ \hat{X}_2$ are constants) and deduce from the equations in (ii) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(\dfrac{m g + k b}{m}\right) & \dfrac{k b}{m} \\ \dfrac{k b}{m} & -\left(\dfrac{m g + k b}{m}\right) \end{pmatrix}$$

(4 marks)

(iv) Find the eigenfrequencies ω of this coupled system and show that they are

$$\omega = \sqrt{\frac{g}{b}} \quad or \quad \sqrt{\frac{m g + 2 k b}{m b}}$$

(7 marks)

Question five

(a) Two sets of coordinate systems are having the same origins. The non-prime system (with position vector denoted as $\vec{r}$ and referred to as "rotating" system) is rotating with an angular velocity $\vec{\omega}$ about the prime system (with position vector denoted as $\vec{r}\,'$ and referred as "fixed" system and taken as an inertial system). Use the following proven relation that $\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \left(\frac{d\vec{F}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{F}$ for any vector field $\vec{F}$ to deduce the following:

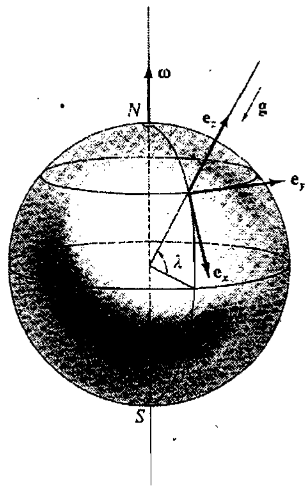
$$\vec{a}_{eff} = \vec{a} - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\,\vec{\omega}\times\vec{v}_r \quad where$$

$$\vec{r}\,' = \vec{r}\ \ (same\ origin)\ ,\ \vec{a}_{eff} \equiv \left(\frac{d^2\vec{r}}{dt^2}\right)_{rotating}\ ,\ \vec{a} \equiv \left(\frac{d^2\vec{r}\,'}{dt^2}\right)_{fixed}\ ,\ \vec{v}_r \equiv \left(\frac{d\vec{r}}{dt}\right)_{rotating}\ ,$$

$$\left(\frac{d\vec{v}_r}{dt}\right)_{rotating} \equiv \left(\frac{d^2\vec{r}}{dt^2}\right)_{rotating} \quad and \quad \dot{\vec{\omega}} \equiv \left(\frac{d\vec{\omega}}{dt}\right)_{rotating}$$

(12 marks)

(b)



Show that the horizontal deflection d along $-\vec{e}_y$ direction resulting from the Coriolis force $(-2\,m\,\vec{\omega}\times\vec{v}_r)$ of a particle falling freely in the earth's gravitational field at a northern latitude λ is

$$d \approx \frac{1}{3}\,\omega\cos(\lambda)\sqrt{\frac{8\,h^3}{g}} \qquad where$$

ω : *angular velocity of earth's rotation* **(13 marks)**

h : *the height of the particle above the earth before its free fall*

(Hint :

$$\vec{a}_{eff} \approx \vec{e}_z\,(-g) - 2\,\vec{\omega}\times\vec{v}_r\ ,\ \vec{v}_r \approx \vec{e}_z\,(-g\,t)\ ,\ \vec{\omega} = \vec{e}_x\,(-\omega\cos(\lambda)) + \vec{e}_z\,(\omega\sin(\lambda))$$

$$and\ \ (total\ time\ for\ the\ given\ motion) = \sqrt{\frac{2\,h}{g}}$$)

Useful informations

$V = -\int \vec{F} \bullet d\vec{l}$ *and reversely* $\vec{F} = -\vec{\nabla} V$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad and \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad and \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

radius of earth $r_E = 6.4 \times 10^6 \ m$

mass of earth $m_E = 6 \times 10^{24} \ kg$

earth attractive potential $\equiv -\dfrac{k}{r}$ *where* $k = G\, m\, m_E$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k}} \quad \{(\varepsilon = 0 \,,\, circle), (0 < \varepsilon < 1 \,,\, ellipse), (\varepsilon = 1 \,,\, parabola), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad if \quad m_2 >> m_1$$

$$For\ elliptical\ orbit, i.e., 0 < \varepsilon < 1, then \begin{cases} semi-major \quad a = \dfrac{k}{2\,|E|} \\ semi-\min or \quad b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ period \quad \tau = \dfrac{2\,\mu}{l} (\pi\, a\, b) \\ r_{\min} = a\,(1 - \varepsilon) \ \&\ r_{\max} = a\,(1 + \varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r \left(\ddot{r} - r\, \dot{\theta}^2 \right) + \vec{e}_\theta \left(2\, \dot{r}\, \dot{\theta} + r\, \ddot{\theta} \right) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,1}\, x_{\alpha,2} & -\sum_\alpha m_\alpha \, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,2}\, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,3}\, x_{\alpha,1} & -\sum_\alpha m_\alpha \, x_{\alpha,3}\, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\,m\,\vec{\omega} \times \vec{v}_r \qquad \textit{where}$$

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$