UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2012/2013

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| TITLE OF PAPER | : | CLASSICAL MECHANICS |
| COURSE NUMBER | : | P320 |
| TIME ALLOWED | : | THREE HOURS |
| INSTRUCTIONS | : | ANSWER ANY FOUR OUT OF FIVE QUESTIONS.<br>EACH QUESTION CARRIES 25 MARKS.<br>MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. |

THIS PAPER HAS TEN PAGES, INCLUDING THIS PAGE.

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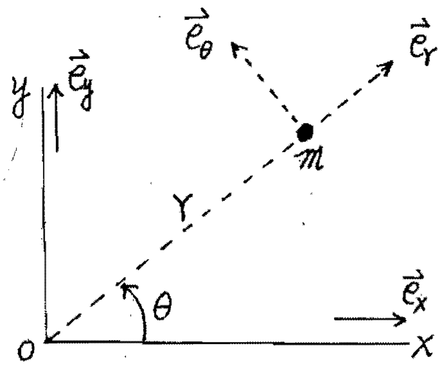
**P320 CLASSICAL MECHANICS**

**Question one**

Consider a particle of mass $m$ acted on by an attractive central force of $\vec{F} = -\ \vec{e}_r \frac{k}{r^5}$ , where $k$ is a positive constant , and moving in a 2-D plane described by the plane polar coordinates as shown in the diagram below.



(a) (i) From $\vec{r} = \vec{e}_r\, r$ and

$\vec{e}_r = \vec{e}_x \cos(\theta) + \vec{e}_y \sin(\theta)$ & $\vec{e}_\theta = -\vec{e}_x \sin(\theta) + \vec{e}_y \cos(\theta)$

(where $\vec{e}_x$ & $\vec{e}_y$ are constant unit vectors while $\vec{e}_r$ & $\vec{e}_\theta$ are not) , deduce that $\vec{v} \equiv \dot{\vec{r}} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta}$ **( 3marks )**

(ii) From $T\left( = \frac{m}{2}(\vec{v} \bullet \vec{v})\right)$ and deduce that the kinetic energy of this particle in this plane polar coordinate is $T = \frac{m}{2}\left(\dot{r}^2 + r^2\, \dot{\theta}^2\right)$. **( 2 marks )**

(ii) From $V = -\int_{r_0}^{r} \vec{F} \bullet d\vec{l}$ *where* $d\vec{l} = d\vec{r} = \vec{e}_r\, d\,r + \vec{e}_\theta\, r\, d\,\theta$ & $r_0 \to \infty$ , find the potential energy $V$ of this particle in this plane polar coordinate under the given force $\vec{F} = -\ \vec{e}_r \frac{k}{r^5}$ where $k$ is a constant. Show that

$V = -\left(\frac{k}{4\,r^4}\right)$ . **( 3 marks )**

(iii) Write down the Lagrange equations of motion for this system and show that

$$\begin{cases} m\,\ddot{r} = \left(m\,r\,\dot{\theta}^2 - \frac{k}{r^5}\right) \\ \frac{d}{dt}\left(m\,r^2\,\dot{\theta}\right) = 0 \end{cases}.$$

**( 6 marks )**

(iv) Write its $(r\,,\theta)$ respective momentums, i.e., $p_r$ & $p_\theta$ . **( 2 marks )** .

**Question one (continued)**

(b) (i) Since the Lagrangian L of the system is not explicitly depending on t, this implies the Hamiltonian H can be simply written as $H = T + V$, deduce that the Hamiltonian $H$ of the system is $H = \left( \frac{(p_r)^2}{2m} + \frac{(p_\theta)^2}{2mr^2} \right) + \left( -\frac{k}{4r^4} \right)$.

**( 3 marks )**

(ii) Write down the Hamilton's equations of motion for this system and show that

$$\begin{cases} \dot{r} = \dfrac{p_r}{m} \\ \dot{\theta} = \dfrac{p_\theta}{m r^2} \\ \dot{p}_r = \dfrac{(p_\theta)^2}{m r^3} - \dfrac{k}{r^5} \\ \dot{p}_\theta = 0 \end{cases}$$

**( 6 marks )**

**Question two**

For a particle of mass $m$ acted on by an earth gravitational force of $\vec{F} = - \ \vec{e}_y \ m \ g$ and undergoing a projectile motion near the earth surface in a x-y plane where x-direction is along the horizontal direction, there is no other force acting on the particle .

(a) Write down the Hamiltonian $H$ of the system , i.e., $H(x, y, p_x, p_y)$ , and show that

$$H = \frac{p_x^2}{2\,m} + \frac{p_y^2}{2\,m} + m\,g\,y$$

**( 5 marks )**

(b) From the definition of the Poisson brackets , i.e., $[F,G] \equiv \sum_{a=1}^{n} \left( \frac{\partial F}{\partial q_a} \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a} \frac{\partial G}{\partial q_a} \right)$ ,

evaluate $[x,H]$ , $[y,H]$ , $[p_x,H]$ *and* $[p_y,H]$ . **( 8 marks )**

(c) For an equation of the type $\frac{d\,u}{d\,t} = [u,H]$ the specific solution of $u(t)$ is given by the following series expansion

$$u(t) = u_0 + [u,H]_0 \; t + [[u,H],H]_0 \; \frac{t^2}{2!} + [[[u,H],H],H]_0 \; \frac{t^3}{3!} + \cdots\cdots$$

where subscript 0 denotes the initial conditions at $t = 0$ .

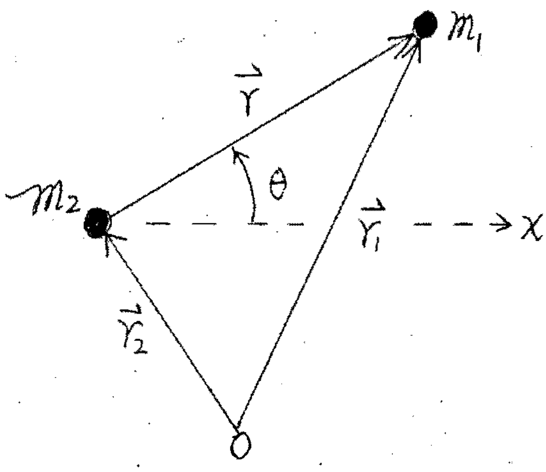
Use the above relation to show that for the given Hamiltonian, the specific solutions of $x(t)$ and $y(t)$ are given by

$$\begin{cases} x(t) = x_0 + \dfrac{p_{x,0}}{m}\, t \\ y(t) = y_0 + \dfrac{p_{y,0}}{m}\, t - \dfrac{g}{2}\, t^2 \end{cases}$$

where $x_0$ *and* $p_{x,0}$ are the initial x-position and x-momentum and $y_0$ *and* $p_{y,0}$ are the initial y-position and y-momentum . **( 12 marks )**

## Question three

(a) A two-body system is depicted below




where $\vec{r}_1$ & $\vec{r}_2$ are the position vectors of $m_1$ & $m_2$ respectively.
Define the center of mass of the system and show that the total kinetic energy of the system , i.e., $T = \frac{1}{2} m_1 \left(\dot{\vec{r}}_1 \bullet \dot{\vec{r}}_1\right) + \frac{1}{2} m_2 \left(\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2\right)$ , can be reduced to $T = \frac{1}{2} \mu \left(\dot{\vec{r}} \bullet \dot{\vec{r}}\right)$ $\left( \textit{where} \quad \mu = \frac{m_1 \, m_2}{m_1 + m_2} \; \textit{is the reduced mass} \right)$ if the center of mass is chosen to be the origin. **( 9 marks )**

(b) If an earth satellite of $500 \; kg$ mass is having a pure tangential speed $v_\theta \left(= r\,\dot{\theta}\right) = 8{,}000 \; \frac{m}{s}$ at its near-earth-point $400 \; km$ above the earth surface,

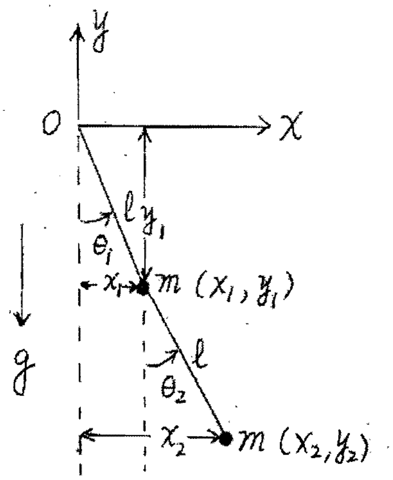
(i) calculate the values of the angular momentum $l$ *and the total energy* $E$ of this satellite , **( 5 marks )**

(ii) calculate the values of the eccentricity $\varepsilon$ and show that the orbit is an elliptical orbit. Also calculate its period. **( 8 marks )**

(iii) What is the value of the pure tangential speed the satellite should have at $400 \; km$ above the earth surface such that its orbit is circular in shape? **( 3 marks )**

**Question four**

Two identical simple pendulums with mass $m$ attached at the end of a massless rod of length $l$ The rod of the second pendulum is attached to the mass of the first pendulum as shown below




The kinetic and potential energies for the system in terms of $x_1$ , $y_1$ , $x_2$ & $y_2$ are

$$T=\frac{1}{2}m\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{1}{2}m\left(\dot{x}_2^2+\dot{y}_2^2\right) \quad \& \quad U=m\,g\,y_1+m\,g\,y_2$$

(i) Show that the Lagrangian for the system in terms of $\theta_1$ *and* $\theta_2$ can be expressed as:

$$L=m\,l^2\,\dot{\theta}_1^2+\frac{1}{2}m\,l^2\,\dot{\theta}_2^2+m\,l^2\,\dot{\theta}_1\,\dot{\theta}_2\cos(\theta_1-\theta_2)+2\,m\,g\,l\cos(\theta_1)+m\,g\,l\cos(\theta_2)$$

**( 10 marks )**

(ii) Write down the equations of motion and deduce that

$$\begin{cases}\ddot{\theta}_1+\frac{1}{2}\left(\ddot{\theta}_2\cos(\theta_1-\theta_2)+\dot{\theta}_2^2\sin(\theta_1-\theta_2)\right)+\frac{g}{l}\sin(\theta_1)=0\\ \ddot{\theta}_2+\left(\ddot{\theta}_1\cos(\theta_1-\theta_2)-\dot{\theta}_1^2\sin(\theta_1-\theta_2)\right)+\frac{g}{l}\sin(\theta_2)=0\end{cases}$$

**( 7 marks )**

(iii) The very rough approximated equations of motion for small $\theta_1$ *and* $\theta_2$ are given below

$$\begin{cases}\ddot{\theta}_1+\frac{1}{2}\ddot{\theta}_2+\frac{g}{l}\theta_1=0\\ \ddot{\theta}_2+\ddot{\theta}_1+\frac{g}{l}\theta_2=0\end{cases}$$

Set $\theta_1=\hat{X}_1\,e^{i\omega t}$ *and* $\theta_2=\hat{X}_2\,e^{i\omega t}$ (where $\hat{X}_1$ *and* $\hat{X}_2$ are constants) and deduce from the given approximated equations the following equations for $\hat{X}_1$ *and* $\hat{X}_2$ as

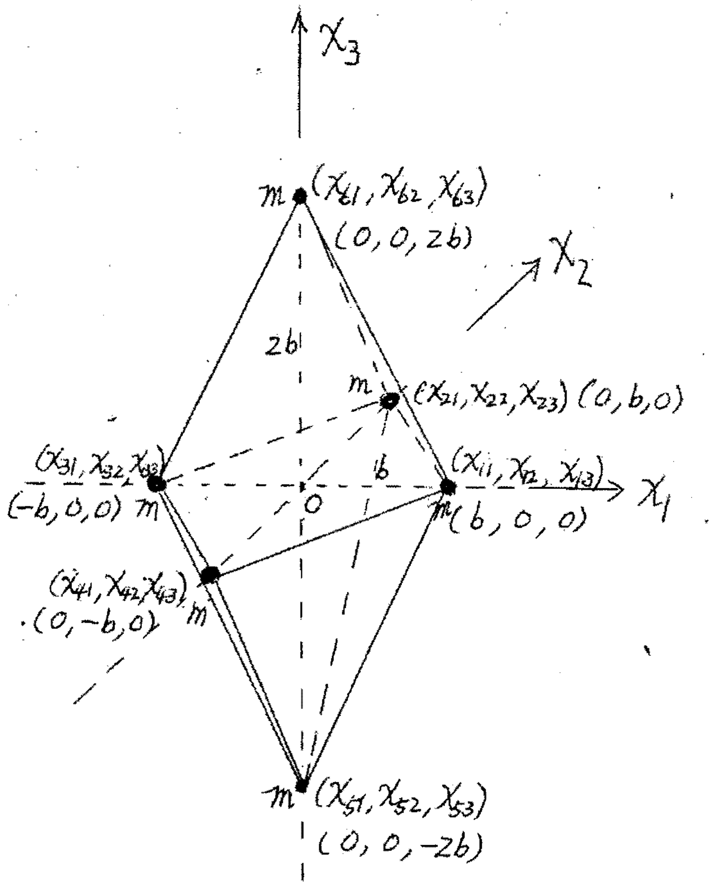
$$\begin{cases}\left(-\omega^2+\frac{g}{l}\right)\hat{X}_1-\frac{1}{2}\omega^2\,\hat{X}_2=0\\ -\omega^2\,\hat{X}_1+\left(-\omega^2+\frac{g}{l}\right)\hat{X}_2=0\end{cases}$$

Then find the appropriate values of $\omega$ such that $\hat{X}_1$ *and* $\hat{X}_2$ can have non-zero solutions.

**( 4+4 marks )**

## Question five

Six equal mass points, each of mass $m$, are attached by massless rigid rods to form a rigid body. The coordinates of each mass point in the body coordinate system $(x_1, x_2, x_3)$ are indicated and its origin is chosen and shown in the diagram below.




The coordinates of each mass point in terms of length $b$ are indicated in the diagram.

(a) Evaluate all elements of the inertia tensor $I$ of the given rigid body with respect to the chosen body coordinate system and show that

$$I = \begin{pmatrix} 10\,m\,b^2 & 0 & 0 \\ 0 & 10\,m\,b^2 & 0 \\ 0 & 0 & 4\,m\,b^2 \end{pmatrix}$$

**( 9 marks )**

(b) If the given rigid body is only rotating with an angular velocity $\vec{\omega}$ without translational motion with respect to a fixed inertia coordinate system $(x'_1, x'_2, x'_3)$, write down the total kinetic energy $T = T_{rotational} = \frac{1}{2}\,\vec{\omega} \bullet I \bullet \vec{\omega}$ in terms of $\omega_1, \omega_2$ & $\omega_3$ where $\vec{\omega} = \vec{e}_1\,\omega_1 + \vec{e}_2\,\omega_2 + \vec{e}_3\,\omega_3$

**( 3 marks )**

## Question five (continued)

(c) The following are Euler's equations for force-free pure-rotational motion, i.e., $L = T_{rotational}$, for already diagonalized $I$ as the case in (a).

$$\begin{cases} (I_2 - I_3)\,\omega_2\,\omega_3 - I_1\,\dot{\omega}_1 = 0 \\ (I_3 - I_1)\,\omega_3\,\omega_1 - I_2\,\dot{\omega}_2 = 0 \\ (I_1 - I_2)\,\omega_1\,\omega_2 - I_3\,\dot{\omega}_3 = 0 \end{cases}$$

(i) For our given rigid body, deduce from the above Euler's equations that

$$\begin{cases} \omega_3 = const. \xrightarrow{set\ as} K \\ \dot{\omega}_1 = \dfrac{3K}{5}\,\omega_2 \quad \cdots\cdots \quad (A) \\ \dot{\omega}_2 = -\dfrac{3K}{5}\,\omega_1 \quad \cdots\cdots \quad (B) \end{cases}$$

**( 5 marks )**

(ii) Deduce from eq.(A) and eq.(B) in (c)(i) that

$$\ddot{\omega}_1 = -\left(\frac{3K}{5}\right)^2 \omega_1 \quad \cdots\cdots \quad (C)$$

**( 1 marks )**

(iii) By direct substitution, show that $\omega_1 = A\cos\left(\dfrac{3K}{5}t + B\right)$ is the the solution to eq.(C) with A & B constant values linking to the given initial value of $\omega_1$ .

**( 3 marks )**

(iv) Substitute $\omega_1 = A\cos\left(\dfrac{3K}{5}t + B\right)$ into eq.(A) in (c)(i) to deduce that $\omega_2 = A\cos\left(\dfrac{3K}{5}t + B + \pi\right)$ , i.e., $\omega_1$ & $\omega_2$ are having the same wave amplitude but totally out of phase.

**( 4 marks )**

## Useful informations

$V = -\int \vec{F} \bullet d\vec{l}$ *and reversely* $\vec{F} = -\vec{\nabla} V$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad and \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad and \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left( \frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

*radius of earth* $r_E = 6.4 \times 10^6 \ m$

*mass of earth* $m_E .= 6 \times 10^{24} \ kg$

*earth attractive potential* $\equiv -\frac{k}{r}$ *where* $k = G\, m\, m_E$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k}} \quad \{(\varepsilon = 0, circle), (0 < \varepsilon < 1, ellipse), (\varepsilon = 1, parabola), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad if \quad m_2 >> m_1$$

$$For\ elliptical\ orbit, i.e., 0 < \varepsilon < 1, then \begin{cases} semi-major \ \ a = \dfrac{k}{2|E|} \\ semi-\min or \ \ b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ period \ \ \tau = \dfrac{2\,\mu}{l}(\pi\, a\, b) \\ r_{\min} = a(1-\varepsilon) \ \& \ r_{\max} = a(1+\varepsilon) \end{cases}$$

*for plane polar* $(r, \theta)$ *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r\, (\ddot{r} - r\, \dot{\theta}^2) + \vec{e}_\theta\, (2\, \dot{r}\, \dot{\theta} + r\, \ddot{\theta}) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

## Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \; x_{\alpha,1} \; x_{\alpha,2} & -\sum_\alpha m_\alpha \; x_{\alpha,1} \; x_{\alpha,3} \\ -\sum_\alpha m_\alpha \; x_{\alpha,2} \; x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \; x_{\alpha,2} \; x_{\alpha,3} \\ -\sum_\alpha m_\alpha \; x_{\alpha,3} \; x_{\alpha,1} & -\sum_\alpha m_\alpha \; x_{\alpha,3} \; x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$