UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2012/2013

TITLE OF PAPER	:	ELECTROMAGNETIC THEORY
COURSE NUMBER	:	P331
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P331 ELECTROMAGNETIC THEORY

Question one

(a) (i) For any closed surface S, enclosing a volume V, the integral form of continuity equation for electric charges in Electromagnetic theory can be written as

$$\oint_S \vec{J} \bullet d\vec{s} = -\frac{d}{dt}\left(\oint_V \rho_v \, dv\right)$$

. Explain briefly the meaning of the left hand side and the right hand side of this equation and indicate which law in physics it describes. **(3+1 marks)**

(ii) Use the divergence theorem to transform the above integral form of continuity equation for electric charges into its differential form. **(3 marks)**

(iii) (A) Show that without introducing the displacement current term, i.e., $\frac{\partial \vec{D}}{\partial t}$, in the equation for Ampere's law, i.e., $\vec{\nabla} \times \vec{H} = \vec{J}$ instead of

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

, Maxwell's equations would contradict the continuity equation for electric charges. **(2 marks)**

(B) Show that by including the displacement current term, Maxwell's equations are in agreement with the continuity equation. **(4 marks)**

(b) (i) From the time-independent Maxwell's equations deduce the following Poison's equation for the electric scalar potential f in free space as

$$\nabla^2 f = -\frac{\rho_v}{\varepsilon_0} \qquad where \quad \vec{E} \equiv -\vec{\nabla} f$$

(3 marks)

(ii) The Pointing vector $\vec{R} \equiv \vec{r} - \vec{r}\,' \equiv \vec{e}_R \, R$ is from the source point $\vec{r}\,' \equiv \vec{e}_x \, x' + \vec{e}_y \, y' + \vec{e}_z \, z'$ toward the field point $\vec{r} \equiv \vec{e}_x \, x + \vec{e}_y \, y + \vec{e}_z \, z$. By direct evaluation of $\vec{\nabla}\left(\frac{1}{R}\right)$ where

$$\begin{cases} R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\ \vec{\nabla} \rightarrow \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \\ \vec{e}_R = \vec{e}_x \frac{(x-x')}{R} + \vec{e}_y \frac{(y-y')}{R} + \vec{e}_z \frac{(z-z')}{R} \end{cases}$$

show that

$$\vec{\nabla}\left(\frac{1}{R}\right) = -\vec{e}_R \frac{1}{R^2} \quad .$$

(6 marks)

(iii) Assuming the solution for the Poison's equation in (b)(i) is

$$f(x,y,z) = \iiint_{source\ points} \frac{\rho_v(x',y',z')}{4\pi\varepsilon_0 R} \, dx' \, dy' \, dz'$$

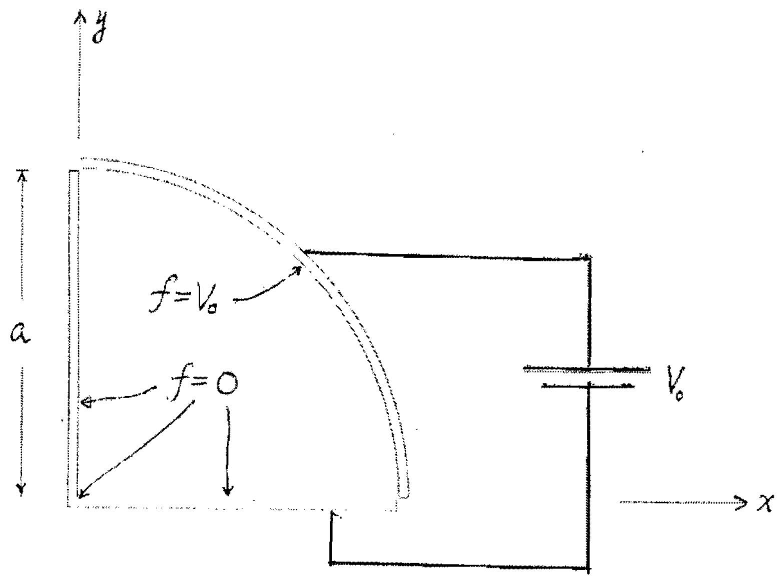
, use the result in (b)(ii) and $\vec{E} \equiv -\vec{\nabla} f$ to deduce that

$$\vec{E}(x,y,z) = \iiint_{source\ points} \left(\vec{e}_R \frac{\rho_v(x',y',z')}{4\pi\varepsilon_0 R^2}\right) dx' \, dy' \, dz'$$

which is just the Coulomb's law. **(3 marks)**

Question two

A V – tube capacitor is extended very long into z direction with its cross section as shown below:



The electric potential $f(\rho,\phi)$ in cylindrical coordinates for the region between two conductors, i.e., $0 \le \rho \le a$ & $0 \le \phi \le \frac{\pi}{2}$, satisfies the following two dimensional Laplace equation :

$$\rho \frac{\partial\left(\rho \frac{\partial f(\rho,\phi)}{\partial \rho}\right)}{\partial \rho} + \frac{\partial^2 f(\rho,\phi)}{\partial \phi^2} = 0$$

(a) (i) Set $f(\rho,\phi) = F(\rho)\, G(\phi)$ and use separation variable scheme to deduce the following two ordinary differential equations :

$$\begin{cases} \rho \dfrac{d\left(\rho \dfrac{d F(\rho)}{d \rho}\right)}{d \rho} = -k\, F(\rho) & \cdots\cdots \quad (1) \\ \dfrac{d^2 G(\phi)}{d \phi^2} = k\, G(\phi) & \cdots\cdots \quad (2) \end{cases}$$

where k is a separation constant of any value. **(4 marks)**

(ii) Based on eq.(2), i.e., differential equation for ϕ , explain why the eigenvalues for k are $k = -m^2$ $\quad where \quad m = 1,2,3,\cdots\cdots$ **(3 marks)**

(iii) By direct substitution, show that ρ^m & ρ^{-m} are the two independent solution to eq.(1) with $k = -m^2$. **(3 marks)**

Question two (continued)

(b) The general solution for (a) is

$$f(\rho,\phi)=\sum_{m=1}^{\infty} f_m(\rho,\phi)$$

$$=\sum_{m=1}^{\infty}\left(A_m\,\rho^{m}+B_m\,\rho^{-m}\right)\left(C_m\cos(m\phi)+D_m\sin(m\phi)\right)\quad\cdots\cdots\quad(3)$$

where A_m , B_m , C_m & D_m are arbitrary constants. This general solution is subjected to the following four boundary conditions :

$BC(1)\ :\ f_m(0,\phi)=0\quad\forall\quad 0\le\phi\le\frac{\pi}{2}$

$BC(2)\ :\ f_m(\rho,0)=0\quad\forall\quad 0\le\rho\le a$

$BC(3)\ :\ f_m(\rho,\frac{\pi}{2})=0\quad\forall\quad 0\le\rho\le a$

$BC(4)\ :\ f(a,\phi)=V_0\quad\forall\quad 0\le\phi\le\frac{\pi}{2}$

(i) Apply BC(1) and deduce from eq.(3) that

$$f(\rho,\phi)=\sum_{m=1}^{\infty}\left(A_m\,\rho^{m}\right)\left(C_m\cos(m\phi)+D_m\sin(m\phi)\right)\quad\cdots\cdots\quad(4)$$

(2 marks)

(ii) Apply BC(2) and deduce from eq.(4) that

$$f(\rho,\phi)=\sum_{m=1}^{\infty}\left(A_m\,\rho^{m}\right)\left(D_m\sin(m\phi)\right)\quad name\quad (A_m\,D_m)\quad as\quad E_m$$

$$=\sum_{m=1}^{\infty}\left(E_m\,\rho^{m}\sin(m\phi)\right)\quad\cdots\cdots\quad(5)$$

(2 marks)

(iii) Apply BC(3) and deduce from eq.(5) that

$$f(\rho,\phi)=\sum_{n=1}^{\infty}\left(F_n\,\rho^{2n}\sin(2n\phi)\right)\quad\cdots\cdots\quad(6)$$

$where\quad F_n\equiv E_{2n}\quad\&\quad n=1,2,3,\cdots\cdots$

(3 marks)

(iv) Apply BC(4) and find the values of F_n in terms of V_0 , a & n and show that

$$F_n=\frac{2\,V_0\,(1-\cos(n\pi))}{n\,\pi\,a^{2n}}\qquad n=1,2,3,\cdots\cdots$$

(8 marks)

(Hint : $\int_{\phi=0}^{\frac{\pi}{2}}\sin(2n\phi)\sin(2m\phi)\,d\phi=\begin{cases}0 & if\quad n\ne m\\ \frac{\pi}{4} & if\quad n=m\end{cases}$)

Question three

(a) The point form of Ohm's law in a conductive region of conductivity σ is $\vec{J} = \sigma \vec{E}$ where $\vec{J}$ & $\vec{E}$ are the current density and electric field respectively. Show that it can lead to the commonly known Ohm's law $V = I R$ for a conducting wire of length L, cross-sectional area A, total flowing current I and the terminal voltage across the wire V *where* $R = \dfrac{L}{\sigma A}$. **(5 marks)**

(b) According to modified Drude's model of electric conduction in the conductive material with conductivity σ under the applied electric field $\vec{E}$, the equation of motion for an average conduction electron in the conductor is

$m_e \dfrac{d\vec{v}_d}{dt} = -e\vec{E} - \dfrac{2 m_e \vec{v}_d}{\tau_c}$ ...... (1) *where* $(-e)$ & m_e are the charge and mass of an electron respectively.

(i) Explain briefly the meaning of $\vec{v}_d$, τ_c & $\left(-\dfrac{2 m_e \vec{v}_d}{\tau_c}\right)$ in the above equation.

(4 marks)

(ii) In the steady state situation, i.e., $\dfrac{d\vec{v}_d}{dt} = 0$, use the equation of motion and the point form of Ohm's law to deduce that

$\sigma = \dfrac{n e^2}{2 m_e} \tau_c$ *where* $n \equiv$ *number density of conduction electrons* .

(Hint : $\vec{J} = \rho_v \vec{v}_d = -n e \vec{v}_d$) **(6 marks)**

(iii) The pure metal potassium K possesses the following data at room temperature as

atomic number $= 39.098$, *density* $= 871 \ \dfrac{kg}{m^3}$ *and*

conductivity $= 1.4 \times 10^7 \ \dfrac{1}{\Omega\, m}$

(A) Calculate the number density of conduction electrons of the metal potassium with the knowledge of each potassium atom contributes one conduction electron. **(3 marks)**

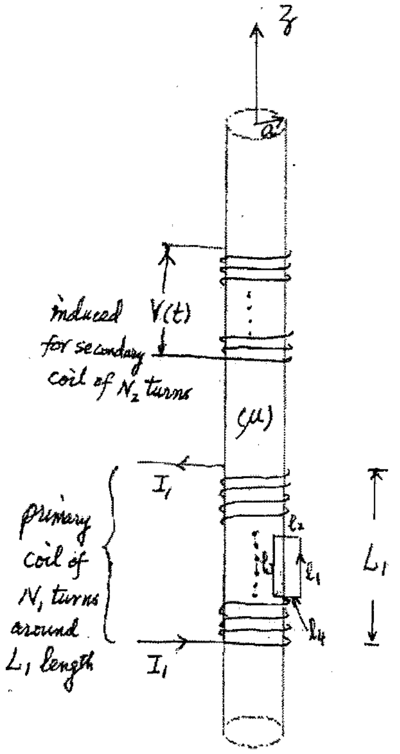
(B) Find the value of τ_c for metal potassium at room temperature.

(Hint : Avogadro number $= 6.023 \times 10^{26} \ \dfrac{atoms}{kg - mole}$) **(3 marks)**

(iv) In the time-harmonic situation, i.e., $\vec{E} = \vec{e}_E\, E_0 \cos(\omega t) = \vec{e}_E\, \mathrm{Re}\{E_0\, e^{i\omega t}\}$, set $\vec{v}_d = \vec{e}_E\, v_d \cos(\omega t + \phi) = \vec{e}_E\, \mathrm{Re}\{\hat{v}_d\, e^{i\omega t}\}$ *where* $\hat{v}_d \equiv v_d\, e^{i\phi}$, use eq. (1) to deduce that $\hat{v}_d = -\dfrac{e\,\tau_c}{2 m_e + i\,\omega\, m_e\, \tau_c} E_0$. **(4 marks)**

Question four

(a) A static current I_1 flows in the N_1 turn toroid wired around an iron core of cross section radius a and permeability μ, with its central axis coinciding with the z – axis as shown below:



(i) Use the closed loop $(l_1 + l_2 + l_3 + l_4)$ drawn in the given diagram where $\vec{l}_1 = \vec{e}_z\, b\,(outside\ \ the\ core), \vec{l}_2 = -\,\vec{e}_\rho\, c\,, \vec{l}_3 = -\,\vec{e}_z\, b\,(inside\ \ the\ core)\,\&\,\vec{l}_4 = \vec{e}_\rho\, c$, set $\vec{B} = \vec{e}_z\, B_z(\rho)\ \ for\ \ \rho \le a\ \ \&\ \ \vec{B} = 0\ \ for\ \ \rho > a$ and use the integral form of Ampere's law to find $\vec{B}$ and show that $\vec{B} = \vec{e}_z\, \dfrac{\mu\, N_1}{L_1}\, I_1\ \ \ for\ \ \ \rho \le a$.

(5 marks)

(ii) Assuming the same $\vec{B}$ obtained in (a)(i) is maintained throughout the iron core (which is a good assumption when $\mu >> \mu_0$), find the total magnetic flux Ψ_m passing through the cross-section area $\pi\, a^2$ of the iron core, i.e.,
$\Psi_m = \int_S \vec{B} \bullet d\vec{s}\ \ \ where\ \ d\vec{s} = \vec{e}_z\ \rho\, d\rho\, d\phi\ \ ,\ \ 0 \le \rho \le a\ \ \&\ \ 0 \le \phi \le 2\pi$, and show that $\Psi_m = \dfrac{\mu\, N_1\, \pi\, a^2}{L_1}\, I_1$.

(2 marks)

Question four (continued)

(iii) Find the mutual inductance M between the primary and secondary coils and the self-inductance L_i of the primary coil in terms of a , L_1 , N_1 , N_2 & μ . **(3 marks)**

(Hint : The total magnetic flux passing through the primary and secondary coils are $N_1\,\Psi_m$ & $N_2\,\Psi_m$ *respectively where* Ψ_m *is obtained in* $(a)(ii)$)

(iv) Find the induced e.m.f. $V(t)$ of the secondary coil in terms of a , L_1 , N_1 , N_2 , μ , I_0 & ω if the primary coil carries a sinusoidal current of $I_0\,\sin(\omega t)$ instead of carrying a static current I_1 . **(2 marks)**

(b) The Maxwell's equations for the material region of parameters μ , ε & σ are

$$\vec{\nabla}\bullet\vec{E}=0 \quad \cdots\cdots \quad (1)$$

$$\vec{\nabla}\bullet\vec{B}=0 \quad \cdots\cdots \quad (2)$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \cdots\cdots \quad (3)$$

$$\vec{\nabla}\times\vec{B}=\mu\,\sigma\,\vec{E}+\mu\,\varepsilon\,\frac{\partial\vec{E}}{\partial t} \quad \cdots\cdots \quad (4)$$

(i) Deduce from them the following wave equation for $\vec{E}$ as

$$\nabla^2\,\vec{E}-\mu\,\sigma\,\frac{\partial\vec{E}}{\partial t}-\mu\,\varepsilon\,\frac{\partial^2\vec{E}}{\partial t^2}=0 \quad \cdots\cdots \quad (5) \ .$$ **(4 marks)**

(ii) Set $\vec{E}$ as $\left(\hat{\vec{E}}(space)\,e^{i\,\omega\,t}\right)$ and substitute it into the above wave equation, to deduce the following time-harmonic equation for $\hat{\vec{E}}(space)$ as

$$\nabla^2\,\hat{\vec{E}}(space)-\hat{\gamma}^2\,\hat{\vec{E}}(space)=0 \qquad where \quad \hat{\gamma}=\sqrt{i\,\omega\,\mu\,\sigma-\omega^2\,\mu\,\varepsilon}$$

(3 marks)

(iii) Set the propagation constant $\hat{\gamma}\equiv\alpha+i\,\beta$, to deduce that

$$\alpha=\frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1} \quad .$$ **(6 marks)**

(Hint : $\sin\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos(\theta)}{2}}$ & $\cos(\theta)=\left(\sqrt{1+\tan^2(\theta)}\right)^{-1}$)

Question five

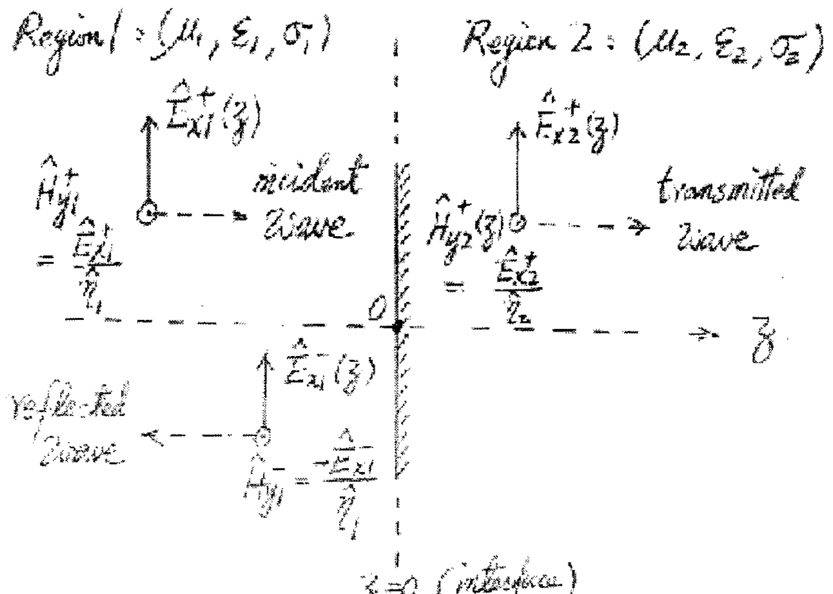
(a) An uniform plane wave traveling along + z direction with the field components $E_x(z)$ & $H_y(z)$ has a complex electric field amplitude $\hat{E}_m^+ = 100\, e^{i\frac{\pi}{6}}\ \frac{V}{m}$ and propagates at a frequency $f = 5 \times 10^7\ Hz$ in a material region has the parameters of $\mu = \mu_0$, $\varepsilon = 2\,\varepsilon_0$ & $\dfrac{\sigma}{\omega\,\varepsilon} = 0.3$.

(i) Find the values of the propagation constant $\hat{\gamma}\ (= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave . **(4 marks)**

(ii) Express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (a)(i) inserted . **(4 marks)**

(iii) Find the values of the penetration depth , wave length and phase velocity of the given wave . **(3 marks)**

(b) An uniform plane wave is incident normally upon an interface separating two regions . The incident wave is given as $\left(\hat{E}_{x1}^+ = \hat{E}_{m1}^+\, e^{-\hat{\gamma}_1 z} ,\ \hat{H}_{y1}^+ = \dfrac{\hat{E}_{m1}^+}{\hat{\eta}_1}\, e^{-\hat{\gamma}_1 z} \right)$ and thus the reflected and transmitted wave can be written as $\left(\hat{E}_{x1}^- = \hat{E}_{m1}^-\, e^{+\hat{\gamma}_1 z} ,\ \hat{H}_{y1}^- = -\dfrac{\hat{E}_{m1}^-}{\hat{\eta}_1}\, e^{+\hat{\gamma}_1 z} \right)$ and $\left(\hat{E}_{x2}^+ = \hat{E}_{m2}^+\, e^{-\hat{\gamma}_2 z} ,\ \hat{H}_{y2}^+ = \dfrac{\hat{E}_{m2}^+}{\hat{\eta}_2}\, e^{-\hat{\gamma}_2 z} \right)$ respectively as shown below:



Question five (continued)

(i) From the boundary conditions at the interface , i.e., both total $\hat{E}_x$ & $\hat{H}_y$ are continuous at z = 0 , deduce the following

$$\begin{cases} \hat{E}^-_{m1} = \hat{E}^+_{m1} \dfrac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \\ \hat{E}^+_{m2} = \hat{E}^+_{m1} \dfrac{2\,\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_1} \end{cases}$$

(9 marks)

(ii) If region1 is air (i.e., $\hat{\eta}_1 = 120\,\pi = 377 \quad \Omega$) , region 2 is a lossy medium with parameters of $\left(\mu_2 = \mu_0 \ , \varepsilon_2 = 9\,\varepsilon_0 \ , \dfrac{\sigma_2}{\omega\,\varepsilon_2} = 1 \right)$, and the incident plane wave is having a complex amplitude of $\hat{E}^+_{m1} = 60\,e^{i\,50^0}\ \dfrac{V}{m}$ and propagates at a frequency of $f = 10^6 \ \ Hz$.

(A) Calculate the value of $\hat{\eta}_2$. **(2 marks)**

(B) Calculate the values of $\hat{E}^+_{m2}$. **(3 marks)**

Useful informations

$$e = 1.6 \times 10^{-19}\ C$$

$$m_e = 9.1 \times 10^{-31}\ kg$$

$$\mu_0 = 4\,\pi \times 10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3 \times 10^{8}\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{\,i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = 120\,\pi\ \ \Omega = 377\ \ \Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E} \bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B} \bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B} \bullet d\vec{s}\right)$$

$$\oint_L \vec{B} \bullet d\vec{l} = \mu \iint_S \vec{J} \bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E} \bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

$$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d\,v \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla}\, f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2\, \vec{F}$$

$$\vec{\nabla}\, f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\, r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\, \rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)}\left(\frac{\partial(F_\phi\, r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\, r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x\, F_x + \vec{e}_y\, F_y + \vec{e}_z\, F_z = \vec{e}_\rho\, F_\rho + \vec{e}_\phi\, F_\phi + \vec{e}_z\, F_z = \vec{e}_r\, F_r + \vec{e}_\theta\, F_\theta + \vec{e}_\phi\, F_\phi$ and

$$d\vec{l} = \vec{e}_x\, d\,x + \vec{e}_y\, d\,y + \vec{e}_z\, d\,z = \vec{e}_\rho\, d\,\rho + \vec{e}_\phi\, \rho\, d\,\phi + \vec{e}_z\, d\,z = \vec{e}_r\, d\,r + \vec{e}_\theta\, r\, d\,\theta + \vec{e}_\phi\, r\sin(\theta)\, d\,\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\, \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$