UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2012/2013

TITLE OF PAPER	:	ELECTROMAGNETIC THEORY
COURSE NUMBER	:	P331
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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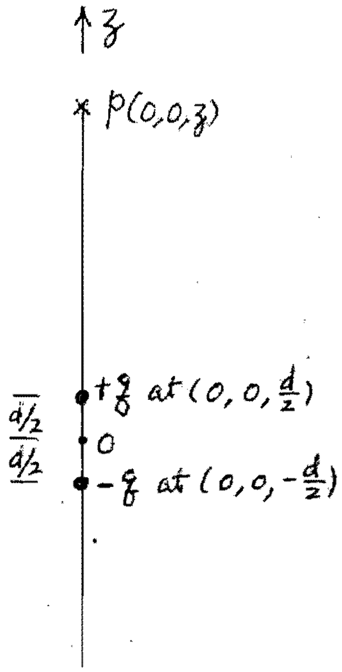
P331 ELECTROMAGNETIC THEORY

Question one

(a) Two equal and opposite point charges $\pm q$ which form an electric dipole with a dipole moment of $\vec{p} = \vec{e}_z\ p = \vec{e}_z\ (q\ d)$, situated at the origin is shown below.



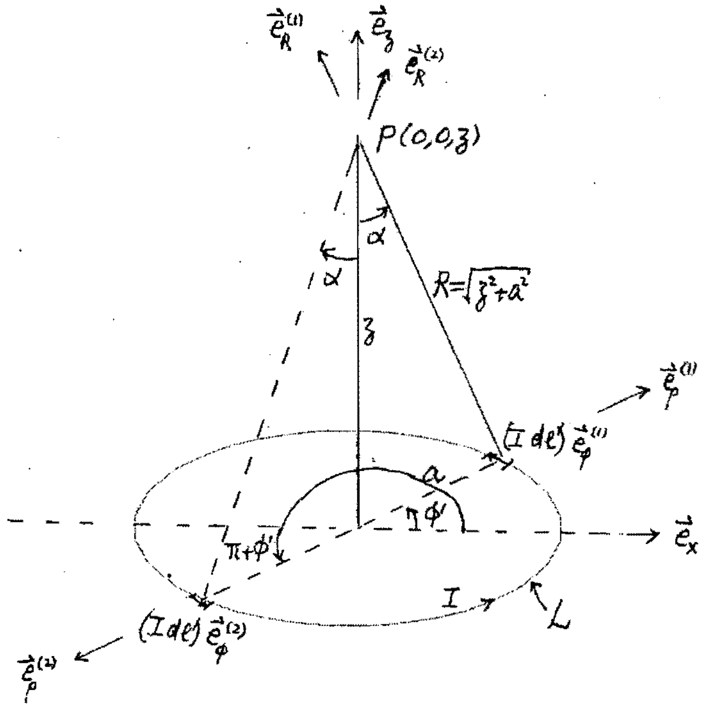
(i) Use superposition principle to write down the electric scalar potential f at the field point P (0,0,z) due to those two point charges which form a dipole. Simplify it and deduce that

$$f = \frac{p}{4\,\pi\,\varepsilon_0 \left(z^2 - \left(\frac{d}{2} \right)^2 \right)} .$$ **(2 marks)**

(ii) Use $\vec{E} \equiv -\vec{\nabla} f$ to find the electric field $\vec{E}$ at the field point P (0,0,z) due to the given dipole. Then show that $\vec{E} \approx \vec{e}_z \dfrac{p}{2\,\pi\,\varepsilon_0\, z^3}$ *if* $z >> d$. **(6 marks)**

Question one (continued)

(b) A circular ring L of radius a carries a constant counter clockwise line current I. The ring is situated on $z = 0$ plane with the ring's centre at the origin as shown in the following diagram



where $\vec{e}_R^{(1)}$ & $\vec{e}_R^{(2)}$ are the unit vectors pointing from the small line segment current sources $(I\,dl')\vec{e}_\phi^{(1)}$ & $(I\,dl')\vec{e}_\phi^{(2)}$ respectively to the field point P (0,0,z).

(i) Express $\vec{e}_R^{(1)}$ & $\vec{e}_R^{(2)}$ in terms of $\vec{e}_\rho^{(1)}$; $\vec{e}_\rho^{(2)}$; $\vec{e}_z$ & α and then show that

$\vec{e}_\phi^{(1)} \times \vec{e}_R^{(1)} + \vec{e}_\phi^{(2)} \times \vec{e}_R^{(2)} = \vec{e}_z\, 2\cos(\alpha)$

(Hint : $\vec{e}_\phi^{(1)} = -\,\vec{e}_\phi^{(2)}$ & $\vec{e}_\rho^{(1)} = -\,\vec{e}_\rho^{(2)}$) **(7 marks)**

(ii) From Biot-Savart law, i.e., $\vec{B} = \int_L \dfrac{\mu_0 \left(\vec{e}_\phi\, I\, dl'\right) \times \vec{e}_R}{4\,\pi\, R^2}$, find the magnetic field $\vec{B}$ at the field point P (0,0,z) produced by the given ring current source. **(6 marks)**

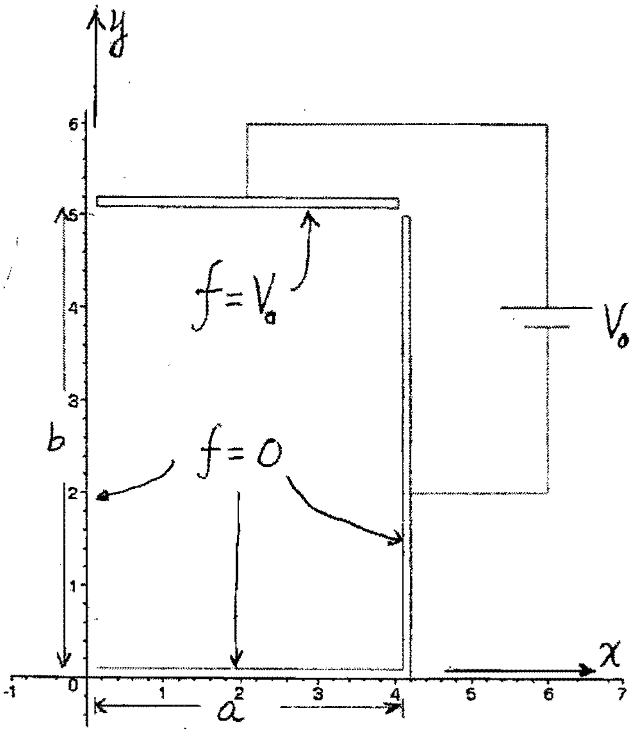
(Hint : use "pair" addition in (b)(i) and then integrate for half of the ring.)

(iii) If $z >> a$, find the approximate expression of $\vec{B}$ from the result obtained in (b)(ii). Rewrite it in terms of magnetic dipole moment of the given current loop m *where* $m \equiv I\left(\pi\, a^2\right)$ and compare it with the approximate expression of $\vec{E}$ in (a)(ii).

Show that replacing $\left(\dfrac{p}{\varepsilon_0}\right)$ by $(\mu_0\, m)$ they are identical. **(4 marks)**

Question two

A U – tube capacitor extended very long into z direction with its cross section as shown below :



Its electric potential $f(x,y)$ in Cartesian coordinates for the region between two conductors, i.e., $0 \le x \le a$ & $0 \le y \le b$, satisfies the following two dimensional Laplace equation :

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0 \ .$$

(a) Set $f(x,y) = F(x)\,G(y)$ and use separation variable scheme

(i) to deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = k\, F(x) & \cdots\cdots \quad (1) \\ \dfrac{d^2 G(y)}{d y^2} = -\,k\, G(y) & \cdots\cdots \quad (2) \end{cases}$$

where k is the separation constant of any value. **(4 marks)**

(ii) Based on the given initial conditions indicated in the above diagram, explain why only the negative values of k are desirable? Thus k can be written as $-K^2$. **(3 marks)**

(iii) By direct substitution, show that $A\cosh(K\,y) + B\sinh(K\,y)$, where A & B are arbitrary constants, is a general solution to eq.(2) with $k = -K^2$ where K is a positive constant and a better alternative for the separation constant. **(3 marks)**

Question two (continued)

(b) The general solution for (a) is

$$f(x,y)=\sum_{\forall K} f_K(x,y)$$

$$=\sum_{\forall K}\left(A_K\cos(Kx)+B_K\sin(Kx)\right)\left(C_K\cosh(Ky)+D_K\sinh(Ky)\right) \quad \cdots\cdots \quad (3)$$

where A_K , B_K , C_K & D_K are arbitrary constants. This general solution is subjected to the following four boundary conditions :

$BC(1)$: $f_K(0,y)=0 \quad \forall \quad 0\le y\le b$

$BC(2)$: $f_K(a,y)=0 \quad \forall \quad 0\le y\le b$

$BC(3)$: $f_K(x,0)=0 \quad \forall \quad 0\le x\le a$

$BC(4)$: $f(x,b)=V_0 \quad \forall \quad 0\le x\le a$

(i) Apply BC(1) and deduce from eq.(3) that

$$f(x,y)=\sum_{\forall K}\left(B_K\sin(Kx)\right)\left(C_K\cosh(Ky)+D_K\sinh(Ky)\right) \quad \cdots\cdots \quad (4)$$

(2 marks)

(ii) Apply BC(3) and deduce from eq.(4) that

$$f(x,y)=\sum_{\forall K}\left(B_K\sin(Kx)\right)\left(D_K\sinh(Ky)\right) \quad name \quad (B_K\ D_K) \quad as \quad E_K$$

$$=\sum_{\forall K}\left(E_K\sin(Kx)\sinh(Ky)\right) \quad \cdots\cdots \quad (5)$$

(2 marks)

(iii) Apply BC(2) and deduce from eq.(5) that

$$f(x,y)=\sum_{n=1}^{\infty}\left(E_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right) \quad \cdots\cdots \quad (6)$$

(3 marks)

(iv) Apply BC(4) and find the values of E_n in terms of V_0 , a & b and show that

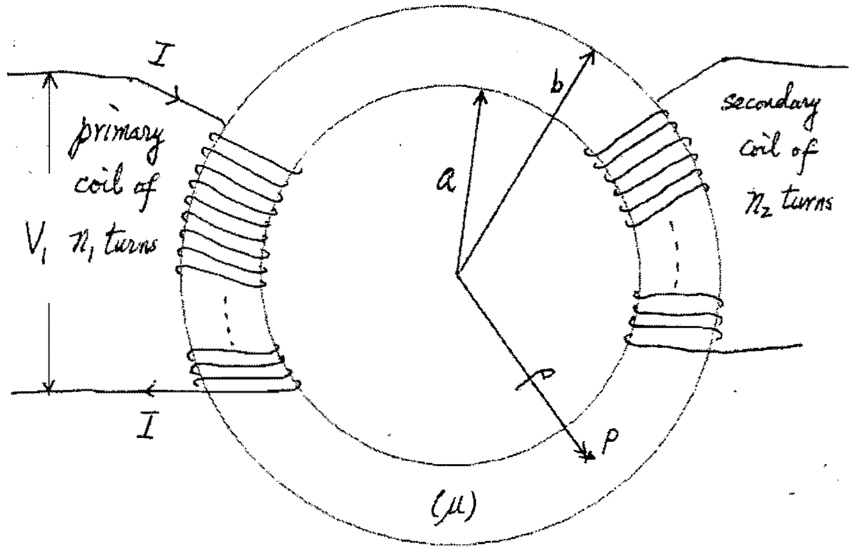
$$E_n=\frac{2V_0\left(1-\cos(n\pi)\right)}{n\pi\sinh\left(\frac{n\pi b}{a}\right)} \qquad n=1,2,3,\cdots\cdots$$

. Also write down the specific solution to this given boundary value problem. **(8 marks)**

(Hint : $\displaystyle\int_{x=0}^{a}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx=\begin{cases}0 & if \quad n\ne m\\ \frac{a}{2} & if \quad n=m\end{cases}$)

Question three

A static current I flows in the primary coil of n_1 turn toroid, wired around an iron ring core of magnetic permeability μ with the square cross-section area $(b-a)^2$ as shown below:



(a) Use the integral Ampere's law , choose and draw proper closed loops to find the magnetic field $\vec{B}$ in terms of ρ , n_1 , μ & I within the iron core , i.e., $a \le \rho \le b$ & $0 \le z \le (b-a)$ region. **(1+6 marks)**

(b) Find the total magnetic flux Ψ_m passing through the cross-section area $(b-a)^2$ of the iron ring in counter clockwise sense, i.e., $\int_S \vec{B} \bullet d\vec{s}$ where $S : a \le \rho \le b$, $0 \le z \le (b-a)$ & $d\vec{s} = \vec{a}_\phi \, d\rho \, dz$, in terms of a, b, n_1, μ & I. **(6 marks)**

(c) Find the self-inductance L of the primary coil as well as the mutual inductance M of the secondary coil due to the primary coil in terms of a, b, μ, n_1 & n_2. **(5 marks)**

(d) (i) If the primary coil carries a sinusoidal current of $I_0 \sin(\omega t)$ instead of carrying a static current I , find the induced e.m.f. $V_2(t)$ for the secondary coil in terms of $a, b, \omega, n_1, n_2, \mu$ & I_0 under quasi static situation . **(4 marks)**

(ii) If the potential drop for the primary coil due to its resistance is negligible compared to the one due to its self-inductance, i.e., $V_1(t) \approx L \dfrac{dI}{dt}$, show that

$$\frac{|V_2(t)|}{|V_1(t)|} = \frac{n_2}{n_1} \; .$$

(3 marks)

Question four

(a) The Maxwell's equations for the empty space are

$\vec{\nabla} \bullet \vec{E} = 0$ ...... (1)

$\vec{\nabla} \bullet \vec{B} = 0$ ...... (2)

$\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ...... (3)

$\vec{\nabla} \times \vec{B} = \mu_0 \, \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ ...... (4)

(i) Deduce from them the following wave equation for $\vec{B}$ as

$\nabla^2 \vec{B} = \mu_0 \, \varepsilon_0 \dfrac{\partial^2 \vec{B}}{\partial t^2}$ ...... (5) . **(4 marks)**

(ii) Given $\vec{B} = \vec{e}_x \, B_m \cos\left(\omega t + \omega \sqrt{\mu_0 \, \varepsilon_0} \, z + \phi\right)$, where B_m , ω & ϕ are constants,

(A) by direct substitution show that it is a solution to eq.(5) in (a)(i). **(4 marks)**

(B) setting its corresponding solution of $\vec{E}$ as

$\vec{E} = \vec{e}_x \, E_x(z,t) + \vec{e}_y \, E_y(z,t) + \vec{e}_z \, E_z(z,t)$, from eq.(3) deduce that

$E_y(z,t) = \left(\dfrac{B_m}{\sqrt{\mu_0 \, \varepsilon_0}}\right) \cos\left(\omega t + \omega \sqrt{\mu_0 \, \varepsilon_0} \, z + \phi\right)$ **(6 marks)**

(b) An uniform plane wave traveling along + z direction with the field components $\hat{E}_x(z)$ & $\hat{H}_y(z)$ has a complex electric field amplitude $\hat{E}_m^+ = 40 \, e^{i\,50^0} \, \dfrac{V}{m}$ and propagates at a frequency $f = 8 \times 10^6 \ Hz$ in a material region having the parameters of $\mu = 8 \, \mu_0$, $\varepsilon = 2 \, \varepsilon_0$ & $\dfrac{\sigma}{\omega \, \varepsilon} = 1$.

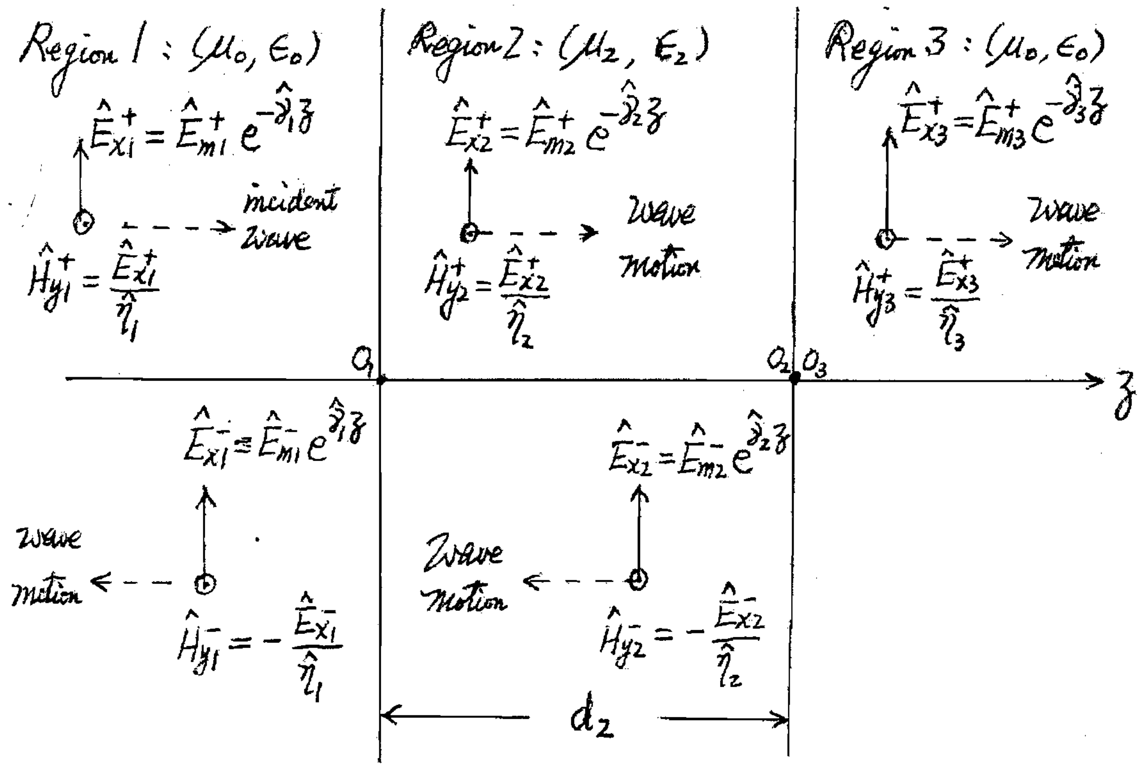
(i) Find the values of the propagation constant $\hat{\gamma} \ (= \alpha + i \, \beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave . **(4 marks)**

(ii) Express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (a)(i) inserted . **(4 marks)**

(iii) Find the values of the penetration depth , wave length and phase velocity of the given wave . **(3 marks)**

Question five

An uniform plane wave $\left(\hat{E}_{x1}^{+}\ ,\ \hat{H}_{y1}^{+}\right)$, operating at $f = 10^{8}\ \ Hz$, is normally incident upon a lossless plate of quarter wavelength thickness, i.e., $d_2 = \dfrac{\lambda_2}{4}$, with parameters of $\left(\mu_2 = \mu_0\ ,\ \varepsilon_2 = 9\ \varepsilon_0\right)$ as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface . (Both region 1 and region 3 are air regions.)

(a) Define for the i^{th} *region* $(i = 1,2,3)$ the reflection coefficient $\hat{\Gamma}_i(z)$ and the total wave impedance $\hat{Z}_i(z)$ and deduce the following :

$$\hat{Z}_i(z) = \hat{\eta}_i\ \frac{1+\hat{\Gamma}_i(z)}{1-\hat{\Gamma}_i(z)}\ .$$

(6 marks)

(b) (i) Find the values of $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, λ_2 & $\hat{\eta}_2$.
(Note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\ \pi\ \ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(4 marks)**

(ii) Starting with $\hat{\Gamma}_3(z) = 0$ for the rightmost region , i.e., region 3 , and using the boundary condition that $\hat{Z}$ is continuous at the interface, find the values of $\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-d_2)$, $\hat{Z}_2(-d_2)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$.

(9 marks)

(iii) Find the value of $\hat{E}_{m1}^{-}$ & $\hat{E}_{m2}^{+}$ *if given* $\hat{E}_{m1}^{+} = 80\ e^{i\,0}\ \dfrac{V}{m}$. **(6 marks)**

Useful informations

$e = 1.6 \times 10^{-19}\ C$

$m_e = 9.1 \times 10^{-31}\ kg$

$\mu_0 = 4\,\pi \times 10^{-7}\ \dfrac{H}{m}$

$\varepsilon_0 = 8.85 \times 10^{-12}\ \dfrac{F}{m}$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3 \times 10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{\,i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}\ .$$

$\eta_0 = 120\,\pi\ \ \Omega = 377\ \ \Omega$

$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B}\bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\oint_L \vec{B}\bullet d\vec{l} = \mu\iint_S \vec{J}\bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E}\bullet d\vec{s}\right)$$

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla}\bullet\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

$$\oint\!\!\oint_S \vec{F} \bullet d\vec{s} \equiv \oint\!\!\oint\!\!\oint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\, r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\, \rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)}\left(\frac{\partial(F_\phi\, r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\, r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z = \vec{e}_\rho F_\rho + \vec{e}_\phi F_\phi + \vec{e}_z F_z = \vec{e}_r F_r + \vec{e}_\theta F_\theta + \vec{e}_\phi F_\phi$ and

$$d\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r\sin(\theta)\, d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

$$\hat{Z}_i(z) = \hat{\eta}_i \frac{1 + \hat{\Gamma}_i(z)}{1 - \hat{\Gamma}_i(z)} \quad , \quad \hat{\Gamma}_i(z) = \frac{\hat{Z}_i(z) - \hat{\eta}_i}{\hat{Z}_i(z) + \hat{\eta}_i} \quad \&$$

$$\hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2\hat{\gamma}_i (z' - z)} \quad \textit{where } z' \;\&\; z \textit{ are two points in } i^{th} \textit{ region}$$