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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

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TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

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Question one

(a) Given a scalar function as $f(r,\theta,\phi) = r^2 - 3\,r^2\,\sin(\theta) + 2\,r^2\,\cos(\phi)$ in spherical coordinates, find the value of $\vec{\nabla} f$ at the point $P:\left(r = 10\,,\theta = \dfrac{\pi}{2}\,,\phi = \dfrac{7\,\pi}{6}\right)$.

(6 marks)

(b) Given a vector field as $\vec{F} = \vec{e}_x\,2\,y^2 + \vec{e}_y\,4\,x\,y - \vec{e}_z\,3\,z^2$ in Cartesian coordinates, find the value of $\int_{P_1,L}^{P_2} \vec{F} \bullet d\vec{l}$ where $P_1 : (-5,-5,1)$ & $P_2 : (5,5,1)$ and if

(i) L : a straight line from P_1 to P_2 on z = 1 plane , **(5 marks)**

(ii) L : a semi-circular path from P_1 to P_2 in counter clockwise sense on z = 1 plane

Compare this answer with that obtained in (b)(i) and comment on the conservative property of the given vector field. **(9 + 1 marks)**

(Hint : circular path with radius of $5\sqrt{2}$ & centered at $(0\,,0\,,1)$, thus $x = 5\sqrt{2}\,\cos(t)$ & $y = 5\sqrt{2}\,\sin(t)$ where t is integrated from $\pi + \dfrac{\pi}{4}$ to $2\,\pi + \dfrac{\pi}{4}$ to follow the counter clockwise sense.

$\int \sin^3(t)\,dt = \frac{1}{3}\cos^3(t) - \cos(t)$ & $\int \sin(t)\cos^2(t)\,dt = -\frac{1}{3}\cos^3(t)$)

(iii) Find $\vec{\nabla} \times \vec{F}$. Does it agree with your comment in (b)(ii)? **(3 + 1 marks)**

Question two

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Given a vector field as $\vec{F} = \vec{e}_\rho \, 3\,\rho\, z + \vec{e}_\phi \, 5\,\rho^2 \sin(\phi) + \vec{e}_z \, 2\,\rho^2$ in cylindrical coordinates,

(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if the closed surface S is the cover surface of a cylinder of radius 5 & height 10 with its central axis coinciding with the z – axis, i.e.,
$S = S_1 + S_2 + S_3$ where
$S_1 \; : \; z = 0\,,\, 0 \le \rho \le 5\,,\, 0 \le \phi \le 2\pi \;\; \& \;\; d\vec{s} = -\,\vec{e}_z\,(d\,\rho)(\rho\, d\,\phi)$
$S_2 \; : \; z = 10\,,\, 0 \le \rho \le 5\,,\, 0 \le \phi \le 2\pi \;\; \& \;\; d\vec{s} = +\,\vec{e}_z\,(d\,\rho)(\rho\, d\,\phi)$
$S_3 \; : \; \rho = 5\,,\, 0 \le \phi \le 2\pi\,,\, 0 \le z \le 10 \;\; \& \;\; d\vec{s} = \vec{e}_\rho\,(\rho\, d\,\phi)(d\,z) \xrightarrow{\rho = 5} \vec{e}_\rho\,(5\, d\,\phi)(d\,z)$

(13 marks)

(b) (i) find $\vec{\nabla} \bullet \vec{F}$, **(5 marks)**

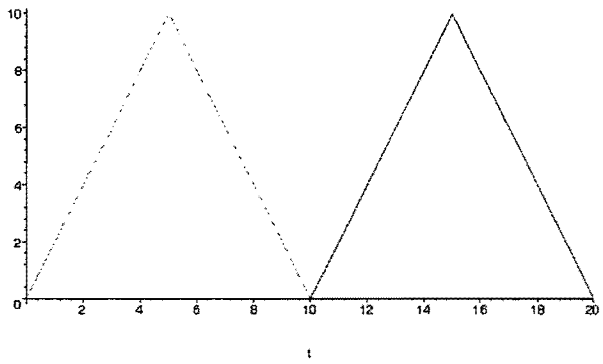
(ii) find the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d\,v$ where
V is the volume enclosed by the given closed surface S described in (b)(i).
Compare this value with the answer obtained in (b)(i) and make a brief comment .

(6 + 1 marks)

(Hint : $d\,v = (d\,\rho)(\rho\, d\,\phi)(d\,z)$ where $0 \le \rho \le 5\,,\, 0 \le \phi \le 2\pi \;\&\; 0 \le z \le 10$)

Question three

Given the following periodic function of period 10 and plotted for two periods from $t = 0$ to $t = 20$ as below :



i.e., for one period of $t = 0$ to $t = 10$, f(t) can be wholly described as :

$$f(t) = \begin{cases} 2t & if \quad 0 < t < 5 \\ -2t + 20 & if \quad 5 < t < 10 \end{cases}$$

(a) Its Fourier series representation is $f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{5}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{5}\right)$.

(i) What special property of the given f(t) implies that all the Fourier sine series coefficients should be zero, i.e., $b_n = 0 \ \forall \ n$. **(2 mark)**

(ii) Find all the Fourier cosine series coefficients and show that

$a_0 = 5$ &

$$a_n = \frac{20\left(\cos(n\pi) - 1\right)}{n^2 \pi^2} \qquad n = 1,2,3,\cdots\cdots$$

(9 marks)

(b) If the above given f(t) is the non-homogeneous term for the following non-homogeneous differential equation $\frac{d^2 y(t)}{dt^2} + 4\,y(t) = f(t)$,

(i) find the particular solution $y_p(t)$ to the given periodical f(t) represented by its Fourier series in (a) and show that

$$y_p(t) = \frac{5}{4} + \sum_{n=1}^{\infty}\left(\frac{500\left(\cos(n\pi) - 1\right)}{n^2 \pi^2 \left(-n^2 \pi^2 + 100\right)} \cos\left(\frac{n\pi t}{5}\right)\right)$$

(11 marks)

(ii) Find the general solution $y_h(t)$ to the homogeneous part of the given differential equation, i.e., $\frac{d^2 y(t)}{dt^2} + 4\,y(t) = 0$, and thus write down the general solution to the non-homogeneous differential equation. **(3 marks)**

Question four

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(a) Given the following first order differential equation as :

$\dfrac{d\,y(x)}{d\,x} + 5\,y(x) = 0$, set $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ & $a_0 \neq 0$ and utilizing the power series method,

(i) write down its indicial equations and show that $s = 0$, **(3 marks)**

(ii) write down its recurrence relation. Set $a_0 = 1$ and use the recurrence relation to find the values of a_1 , a_2 & a_3 and thus write down its independent solution in power series form truncated up to a_3 term. **(8 marks)**

(b) An elastic string of length 10 is fixed at its two ends, i.e., at $x = 0$ & $x = 10$, and its transverse deflection $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\,\frac{\partial^2 u(x,t)}{\partial x^2} ,$$

(i) by direct substitution, show that $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\dfrac{n\pi x}{10}\right) \cos\left(\dfrac{3\,n\pi t}{10}\right)$ satisfies the fixed end conditions $u(0,t) = 0 = u(10,t)$ as well as the condition of $\left.\dfrac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$ assuming there is no initial vibrating speed. **(4 marks)**

(ii) Furthermore, if the initial position of the string , i.e., $u(x,0)$, is given as

$$u(x,0) = \begin{cases} x & if \quad 0 \le x \le 5 \\ -x+10 & if \quad 5 \le x \le 10 \end{cases}$$

, and deduce that

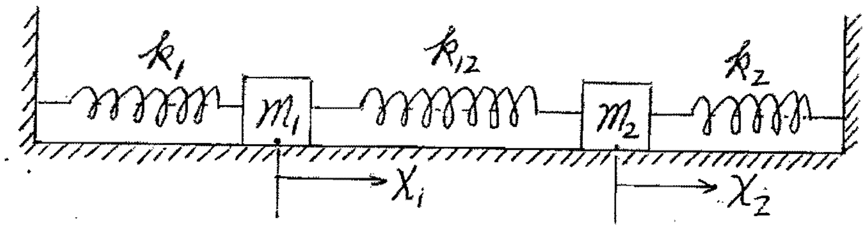
$$E_n = \frac{40}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \qquad \forall \quad n = 1,2,3,\cdots\cdots$$ **(10 marks)**

(hint : $\displaystyle\int_{x=0}^{10} \sin\left(\frac{n\pi x}{10}\right) \sin\left(\frac{m\pi x}{10}\right) d\,x = \begin{cases} 0 & if \quad n \neq m \\ 5 & if \quad n = m \end{cases}$ &

$\displaystyle\int x \sin\left(\frac{n\pi x}{10}\right) d\,x = \frac{100}{n^2\pi^2} \sin\left(\frac{n\pi x}{10}\right) - \frac{10}{n\pi}\, x \cos\left(\frac{n\pi x}{10}\right)$)

Question five

Two simple harmonic oscillators are joined by a spring with a spring constant k_{12} as shown in the diagram below :



The equations of motion for this coupled oscillator system ignoring friction are given as

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -(k_1 + k_{12})\, x_1(t) + k_{12}\, x_2(t) \quad \cdots\cdots \quad (1) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = k_{12}\, x_1(t) - (k_2 + k_{12})\, x_2(t) \quad \cdots\cdots \quad (2) \end{cases}$$

where x_1 & x_2 are horizontal displacements of m_1 & m_2 measured from their respective resting positions.

If given $m_1 = 1\ kg$, $m_2 = 2\ kg$, $k_1 = 2\ \frac{N}{m}$, $k_2 = 4\ \frac{N}{m}$ & $k_{12} = 6\ \frac{N}{m}$,

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, then the above given equations can be deduced to the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -8 & 6 \\ 3 & -5 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(5 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(5 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b), **(6 marks)**

(d) find the normal coordinates of the given coupled system , **(6 marks)**

(e) write down the general solutions for $x_1(t)$ & $x_2(t)$. **(3 marks)**

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Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$

$$f(t) = f(t+2L) = f(t+4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \textit{where}$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_0 = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \quad \textit{for}\; n = 1,2,3$$