UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2013/2014

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P320 CLASSICAL MECHANICS

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Question one

(a) If H denotes the Hamiltonian function and L is the Lagrangian function, use the definition $H = \sum_{\alpha=1}^{n} (p_\alpha \dot{q}_\alpha) - L$ (where p_α *and* q_α $(\alpha = 1, 2, \cdots, n)$ are the generalized momenta and position coordinates respectively, i.e., $H = H(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$, $L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$, $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$ *and* $\frac{\partial L}{\partial q_\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)$) to show that

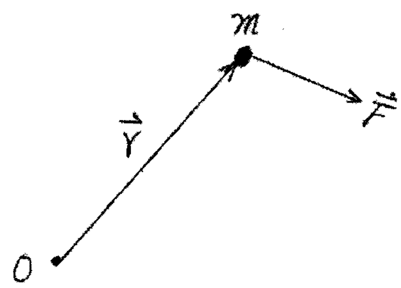
(i) $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$. **(5 marks)**

(ii) Show that for a function $u = u(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$,

$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$ where $[u, H]$ is the Poisson Bracket of two functions u & H , i.e., $[u, H] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right)$. **(4 marks)**

(Hint : $\frac{dq_\alpha}{dt} = \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$ & $\frac{dp_\alpha}{dt} = \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$)

(b) For a particle of mass m in space with a position vector $\vec{r} = \vec{e}_x\, x + \vec{e}_y\, y + \vec{e}_z\, z$ in Cartesian coordinates acted by a force $\vec{F} = \vec{e}_x\, F_x + \vec{e}_y\, F_y + \vec{e}_z\, F_z$ as shown below :



The momentum of m is $\vec{p} = \vec{e}_x\, p_x + \vec{e}_y\, p_y + \vec{e}_z\, p_z$.

(i) Define the angular momentum of m as $\vec{l} \equiv \vec{r} \times \vec{p}$, deduce that

$l_x = y\, p_z - z\, p_y$, $l_y = z\, p_x - x\, p_z$ *and* $l_z = x\, p_y - y\, p_x$ **(3 marks)**

(ii) Define the "moment of force" (or "torque") acted on m as $\vec{N} \equiv \vec{r} \times \vec{F}$, from $\vec{F} = \dot{\vec{p}}$ deduce that $\vec{N} \equiv \dot{\vec{l}}$. **(7 marks)**

(iii) Show that $[l_x, l_z] = -l_y$ and $[y, l_x] = -z$ **(6 marks)**

Question two

A particle of mass m is constrained to move on the surface of a spherical ball with a radius R and centered at the origin. The particle is acted by a conservative force $\vec{F} = -\vec{e}_\theta\, k$. where k is a positive constant.

(a) (i) Use $V = -\int_{P_0}^{P} \vec{F} \bullet d\vec{l}$ *where* $P_0 : (R, 0, 0)$ & $P : (R, \theta, \phi)$ to deduce that $V = k\, R\, \theta$. **(3 marks)**

(Hint : $d\vec{l} \xrightarrow{on\ r=R\ surface} \vec{e}_r\,(0) + \vec{e}_\theta\,(R\, d\theta) + \vec{e}_r\,(R \sin(\theta)\, d\phi)$)

(ii) Write down the Lagrangian of the system in terms of θ & ϕ and then write down their respective equations of motion and deduce that

$$\begin{cases} \dfrac{d}{dt}\left(m R^2\, \dot{\theta}\right) = m R^2 \sin(\theta)\cos(\theta)\, \dot{\phi}^2 - k\, R \\ \dfrac{d}{dt}\left(m R^2 \sin^2(\theta)\, \dot{\phi}\right) = 0 \end{cases}$$

(6 marks)

(Hint : $\vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} + \vec{e}_\phi\, r \sin(\theta)\, \dot{\phi} \xrightarrow{r=R} \vec{e}_\theta\, R\, \dot{\theta} + \vec{e}_\phi\, R \sin(\theta)\, \dot{\phi}$)

(iii) Write down their canonical momenta p_θ & p_ϕ and show that p_ϕ is a constant. **(3 marks)**

(iv) Rewrite the equation of motion for θ in terms of A and deduce that

$$\ddot{\theta} = \frac{A^2 \cos(\theta)}{m^2\, R^4 \sin^3(\theta)} - \frac{k}{m\, R}$$

(4 marks)

(b) (i) One can use $H = T + V$ instead of the definition $H = \sum_{\forall \alpha} \left(p_\alpha\, \dot{q}_\alpha\right) - L$ (make a brief justification for this selection) to write down the Hamiltonian of the system and deduce that

$$H = \frac{(p_\theta)^2}{2\, m\, R^2} + \frac{(p_\phi)^2}{2\, m\, R^2 \sin^2(\theta)} + k\, R\, \theta$$

(1+3 marks)

(ii) From the Hamiltonian in (b)(i), write down the equations of motion of the system. **(5 marks)**

Question three

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(a) Given the Lagrangian for the two-body central force system as :

$$L = T - V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + \frac{k}{r}$$

where μ is the reduced mass of the system, k is a positive constant and (r,θ) are polar coordinates of the motion plane with its origin at the center of mass of the two-body system.

(i) Write down the Lagrange's equation for θ and show that the angular momentum l is conserved , i.e., deduce that

$$\dot{\theta} = \frac{l}{\mu\, r^2} \quad \cdots\cdots \; (1)$$ where l is a constant. **(3 marks)**

(ii) Write down the Lagrange's equation for r , with eq.(1) inserted and deduce that

$$\mu\,\ddot{r} - \frac{l^2}{\mu\, r^3} + \frac{k}{r^2} = 0 \quad \cdots\cdots \; (2)$$ **(3 marks)**

(iii) Multiply eq.(2) by $d\,r$ and use $\ddot{r}\,d\,r = \frac{d\,\dot{r}}{d\,t}\,d\,r = d\,\dot{r}\,\frac{d\,r}{d\,t} = \dot{r}\,d\,\dot{r} = d\left(\frac{\dot{r}^2}{2}\right)$ to show that the total energy $E\,(\equiv T + V)$ is conserved , i.e.,

$$\frac{1}{2}\mu\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) - \frac{k}{r} = const. \equiv E \quad \cdots\cdots \; (3)$$

Also show that eq.(3) can be rewritten as

$$\dot{r} = \sqrt{\frac{2\,E}{\mu} - \frac{l^2}{\mu^2\,r^2} + \frac{2\,k}{\mu\,r}} \quad \cdots\cdots \; (4)$$ **(6 marks)**

(iv) Dividing eq.(1) by eq.(4), deduce the following integral form of orbital equation as

$$\theta = \int \frac{\left(\frac{l}{r^2}\right)}{\sqrt{2\,\mu\left(E - \frac{l^2}{2\,\mu\,r^2} + \frac{k}{r}\right)}}\,d\,r + const. \quad \cdots\cdots \; (5)$$ **(3 marks)**

(b) If an earth satellite of 300 kg mass is having a pure tangential speed $v_\theta\left(= r\,\dot{\theta}\right) = 9{,}000\ \frac{m}{s}$ at its near-earth-point 500 km above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(4 marks)**

(ii) calculate the values of the eccentricity ε and show that the orbit is an elliptical orbit. Also calculate its period. **(6 marks)**

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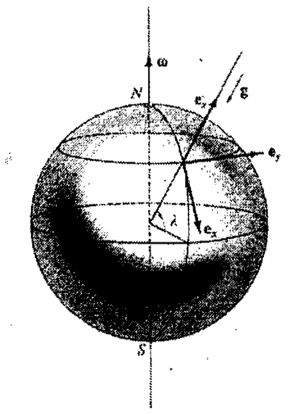
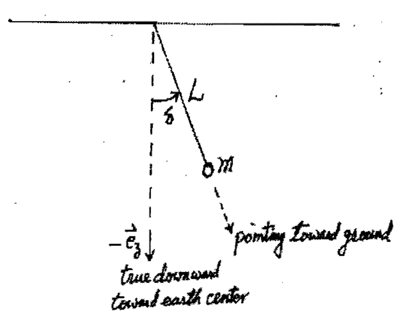
Question four



(a) If a person, standing on the earth surface at a northern latitude λ, fired a bullet of speed v_0 at a target situated at his north direction (i.e., $-\vec{e}_x$ direction) of distance L away from him. Assuming he has a perfect rifle and the time T for the bullet hitting the target is short and $T \approx \frac{L}{v_0}$ (i.e., neglecting the gravitational bending and assuming the bullet is moving along -x direction with constant speed v_0). Show that the bullet will miss the target by a distance d along -y direction resulting from the Coriolis force $(-2\,m\,\vec{\omega}\times\vec{v}_r)$. Show that $d = \frac{\omega L^2}{2 v_0}\sin(\lambda)$. **(10 marks)**

(Hint : $\vec{a}_{eff} \approx -2\,\vec{\omega}\times\vec{v}_r$, $\vec{v}_r \approx \vec{e}_x(-v_0)$, $\vec{\omega} = \vec{e}_x(-\omega\cos(\lambda)) + \vec{e}_z(\omega\sin(\lambda))$)

(b) Refer to the diagram above and consider the body coordinate system (x, y, z) to have the same origin as the earth's fixed inertial system , i.e., center of the earth . If a motionless simple pendulum of length L and mass m is hung near the earth surface at a northern latitude λ , the pendulum suppose to point directly downward along $-\vec{e}_z$ direction. Show that the pendulum is pointing toward a direction not exactly along $-\vec{e}_z$ direction, i.e., true downward direction pointing toward the earth center , but pointing toward the ground with a small angular deviation of δ made with the true downward direction, as shown in the figure below, resulting from the centrifugal force $(-m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}))$. Show that $\delta \approx \frac{\omega^2 r_E \cos(\lambda)\sin(\lambda)}{g - \omega^2 r_E \cos^2(\lambda)}$.

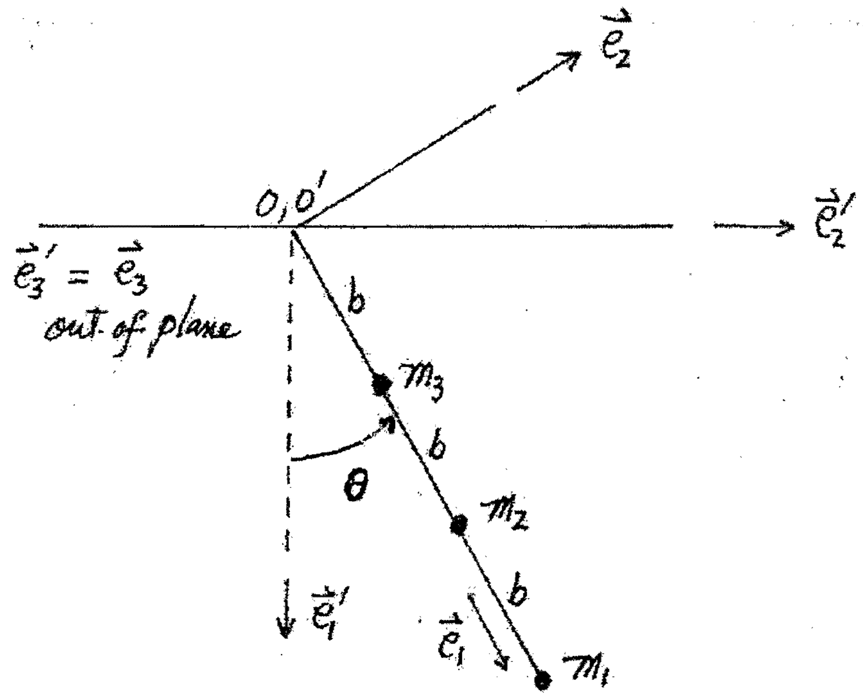


(15 marks)

(Hint : $\vec{F}_{eff} \approx \vec{e}_z(-m\,g) - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\vec{r} \approx \vec{e}_z(r_E)$, $\vec{\omega} = \vec{e}_x(-\omega\cos(\lambda)) + \vec{e}_z(\omega\sin(\lambda))$)

Question five

A pendulum is composed of a rigid rod of length $3\,b$ with a mass m_1 at its end. The second mass m_2 is placed one-third way down the rod while the third mass m_3 is placed two-third way down the rod and the mass of the rod itself is negligible. Let the fixed coordinate system and the body coordinate systems have the same origin at the pendulum pivot point. Let $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$ *and* $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the unit vectors of the fixed (i.e., inertial system) and body coordinate system respectively as shown in the figure below.



(a) Write down the inertia tensor I for the pendulum with respect to the body coordinate system given above and deduce that I is a diagonal matrix with its diagonal elements as $I_{1,1} = 0$ *and* $I_{2,2} = I_{3,3} = (9\,m_1 + 4\,m_2 + m_3)\,b^2$.

(7 marks)

(b) The torque on the pendulum is $\vec{N} = \sum_{\alpha=1}^{3} (\vec{r}_\alpha \times \vec{F}_\alpha) = \vec{e}_1{}' N_1 + \vec{e}_2{}' N_2 + \vec{e}_3{}' N_3$, find N_1 , N_2 & N_3 in terms of θ and show that

$N_1 = 0 = N_2$ & $N_3 = -\,b\,g\sin(\theta)\,(3m_1 + 2m_2 + m_3)$. **(5 marks)**

(*Note*: $\vec{\omega} = \vec{e}_3{}'\dot{\theta}$, $\vec{F}_1 = \vec{e}_1{}' m_1\,g$, $\vec{F}_2 = \vec{e}_1{}' m_2\,g$, $\vec{F}_3 = \vec{e}_1{}' m_3\,g$,
$\vec{r}_1 = \vec{e}_1{}' 3\,b\cos(\theta) + \vec{e}_2{}' 3\,b\sin(\theta)$, $\vec{r}_2 = \vec{e}_1{}' 2\,b\cos(\theta) + \vec{e}_2{}' 2\,b\sin(\theta)$ *and*
$\vec{r}_3 = \vec{e}_1{}' b\cos(\theta) + \vec{e}_2{}' b\sin(\theta)$)

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(c) The following are Euler's equations for pure-rotational motion referring to the inertia coordinate system for already diagonalized I.

$$\begin{cases} (I_2 - I_3)\,\omega_2\,\omega_3 - I_1\,\dot{\omega}_1 = N_1 & \cdots\cdots \quad (1) \\ (I_3 - I_1)\,\omega_3\,\omega_1 - I_2\,\dot{\omega}_2 = N_2 & \cdots\cdots \quad (2) \\ (I_1 - I_2)\,\omega_1\,\omega_2 - I_3\,\dot{\omega}_3 = N_3 & \cdots\cdots \quad (3) \end{cases}$$

where $I_{1,1} \to I_1$, $I_{2,2} \to I_2$ & $I_{3,3} \to I_3$

(i) Insert the results of (a) & (b) for our given rigid body system into the above Euler's equation and deduce that for small θ oscillation, i.e., $\sin(\theta) \approx \theta$,

$$\ddot{\theta} \approx -\left(\frac{g\,(3\,m_1 + 2\,m_2 + m_3)}{b\,(9\,m_1 + 4\,m_2 + m_3)}\right)\theta \quad \cdots\cdots \quad (4)$$

(4 marks)

(ii) By direct substitution, show that $\theta = A\cos(\omega_0\,t + B)$ is the general solution to eq.(4) with A & B are arbitrary constants and $\omega_0 = \sqrt{\dfrac{g\,(3\,m_1 + 2\,m_2 + m_3)}{b\,(9\,m_1 + 4\,m_2 + m_3)}}$.

(4 marks)

(iii) If given the values $m_1 = m_2 = m_3 = 1\ kg$ & $b = 0.1\ m$ and given the initial conditions $\theta(0) = \dfrac{\pi}{10}$ & $\dot{\theta}(0) = 0$, determine the values of ω_0 , A & B and thus write down the specific solution of the given system. **(5 marks)**

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad \text{and reversely} \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \text{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

radius of earth $r_E = 6.4 \times 10^6 \ m$

mass of earth $m_E = 6 \times 10^{24} \ kg$

earth attractive potential $\equiv -\frac{k}{r}$ *where* $k = G\, m\, m_E$

$$\varepsilon = \sqrt{1 + \frac{2 E l^2}{\mu k}} \quad \{(\varepsilon = 0, \text{circle}), (0 < \varepsilon < 1, \text{ellipse}), (\varepsilon = 1, \text{parabola}), \cdots\}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_2 >> m_1$$

$$\text{For elliptical orbit, i.e., } 0 < \varepsilon < 1, \text{ then} \begin{cases} \text{semi-major} \ a = \dfrac{k}{2|E|} \\ \text{semi-minor} \ b = \dfrac{l}{\sqrt{2\mu |E|}} \\ \text{period} \ \tau = \dfrac{2\mu}{l} (\pi\, a\, b) \\ r_{\min} = a(1-\varepsilon) \ \& \ r_{\max} = a(1+\varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r \, \dot{r} + \vec{e}_\theta \, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r \left(\ddot{r} - r\, \dot{\theta}^2\right) + \vec{e}_\theta \left(2\, \dot{r}\, \dot{\theta} + r\, \ddot{\theta}\right) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,2} & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,1} & -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_r$