UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2013/2014

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P320 CLASSICAL MECHANICS

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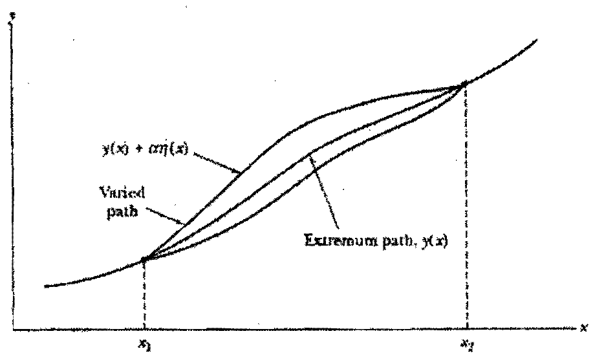
Question one

(a) Given the following definite integral of $J(\alpha)=\int_{x_1}^{x_2} f(y(\alpha,x),y'(\alpha,x),y''(\alpha,x),y'''(\alpha,x);x)\,dx$, where the varied integration path is $y(\alpha,x)=y(x)+\alpha\,\eta(x)$, $\eta(x_1)=\eta(x_2)=0$,

$$\left.\frac{d\eta(x)}{dx}\right|_{x=x_1}=\left.\frac{d\eta(x)}{dx}\right|_{x=x_2}=0 \ \ \& \ \ \left.\frac{d^2\eta(x)}{dx^2}\right|_{x=x_1}=\left.\frac{d^2\eta(x)}{dx^2}\right|_{x=x_2}=0$$ as shown in the following diagram :



Using the extremum condition for $J(\alpha)$, i.e., $\left.\frac{\partial J(\alpha)}{\partial\alpha}\right|_{\alpha=0}=0$, to deduce that

f along the extremum path ,i.e., $f(y(x),y'(x),y''(x),y'''(x);x)$, satisfies the following equation:

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)-\frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right)=0$$ **(12 marks)**

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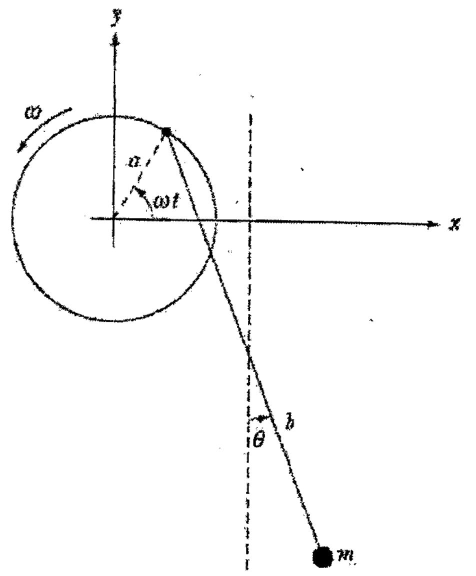
(b) A simple pendulum of length b and mass m moves on a mass-less rim of radius a rotating with constant angular velocity ω as shown in the figure below:



Write down the Lagrangian of the system in terms of θ and then deduce the following equation of motion

$$\ddot{\theta} - \frac{a}{b}\omega^2 \cos(\theta - \omega t) + \frac{g}{b}\sin(\theta) = 0$$

(13 marks)

Question two

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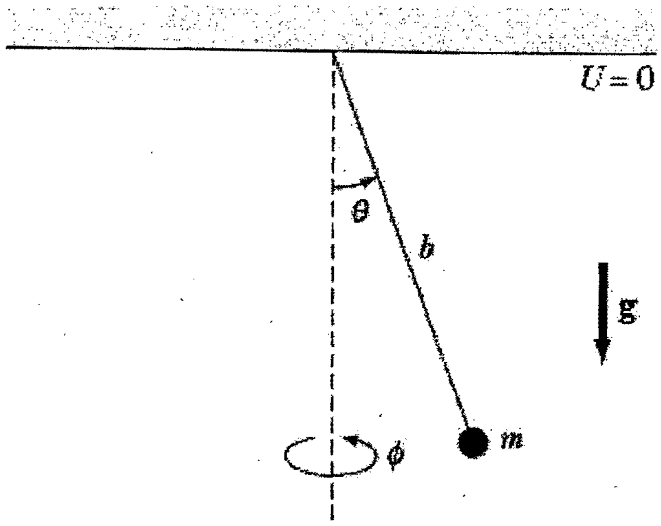
A spherical pendulum of mass m and length b is shown in the figure below:



(a) (i) From $x = b\sin(\theta)\cos(\phi)$, $y = b\sin(\theta)\sin(\phi)$ & $z = -b\cos(\theta)$ and $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ & $V = m\,g\,z$, deduce the following Lagrangian for the system in terms of θ & ϕ as

$$L = \frac{1}{2}m\,b^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)\right) + m\,g\,b\cos(\theta) \quad \cdots\cdots \quad (1)$$

(5 marks)

(ii) Write down the equations of motion for θ & ϕ and deduce that

$$\begin{cases} \dfrac{d\,p_\theta}{d\,t} = m\,b^2\sin(\theta)\cos(\theta)\,\dot{\phi}^2 - m\,g\,b\sin(\theta) & \cdots\cdots \quad (2) \\ \dfrac{d\,p_\phi}{d\,t} = 0 & \cdots\cdots \quad (3) \end{cases}$$

where $p_\theta = m\,b^2\,\dot{\theta}$ & $p_\phi = m\,b^2\sin^2(\theta)\,\dot{\phi}$

(5 marks)

(iii) From eq.(3), one has $p_\phi = const. \xrightarrow{set\ as} K$, deduce from eq.(2) the following equation for small θ , i.e., $\left(\sin(\theta) \approx \theta \;\; and \;\; \cos(\theta) \approx 1 - \frac{\theta^2}{2} \;\; or \;\; 1\right)$, that

$$m^2\,b^4\,\theta^3\,\ddot{\theta} = K^2 - m^2\,g\,b^3\,\theta^4 \quad \cdots\cdots \quad (4)$$

(4 marks)

(iv) If $K = 0$ in eq.(4), write down the general solution of $\theta(t)$. **(3 marks)**

(b) (i) Find the Hamiltonian of the system in terms of θ , ϕ , p_θ & p_ϕ .

(4 marks)

(ii) Write down the equations of motion for H in (b)(i). **(4 marks)**

Question three

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(a) Given the Lagrangian for the two-body central force system as :

$$L = T - V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}$$

where μ is the reduced mass of the system, k is a positive constant and (r,θ) are polar coordinates of the motion plane with its origin at the center of mass of the two-body system. The integral form of orbital equation can be written as

$$\theta = \int \frac{\left(\frac{l}{r^2}\right)}{\sqrt{2\mu\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}} dr + const. \quad \cdots\cdots \quad (1)$$

where $l = \mu r^2 \dot{\theta}$ (i.e., angular momentum) and $E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r}$ (i.e., total energy) are two constants of the system.

Choose the integration constant in eq.(1) as zero (i.e., this is the same as choosing the initial value of r as $r_{\min}$ at $\theta = 0$), and set $u \equiv \frac{1}{r}$,

(i) show that eq.(1) can be simplified as

$$\theta = -\int \frac{1}{\sqrt{\left(\frac{2\mu E}{l^2} - u^2 + \frac{2\mu k}{l^2}u\right)}} du \quad \cdots\cdots \quad (2)$$

(2 marks)

(ii) combine $\left(u^2 - \frac{2\mu k}{l^2}u\right)$ in eq.(2) into the first two terms of a perfect square of u' and show that eq.(2) can be further simplied to

$$\theta = -\int \frac{1}{\sqrt{\left(a^2 - (u')^2\right)}} du' \quad \cdots\cdots \quad (3)$$

(2 marks)

$$where \quad a = \sqrt{\frac{\mu^2 k^2}{l^4} + \frac{2\mu E}{l^2}} \quad \& \quad u' \equiv u - \frac{\mu k}{l^2}$$

(iii) set $u' = a\cos(\beta)$ and carry out the integral of $\int \frac{1}{\sqrt{\left(a^2 - (u')^2\right)}} du'$ and show that eq.(3) becomes $\theta = \beta \quad \cdots\cdots \quad (4)$ **(2 marks)**

(iv) Taking cosine of eq.(4) and using $u' = a\cos(\beta)$, $u' \equiv u - \frac{\mu k}{l^2}$ & $u = \frac{1}{r}$, deduce the following orbital equation

$$\frac{\alpha}{r} \equiv 1 + \varepsilon\cos(\theta) \quad where \quad \alpha \equiv \frac{l^2}{\mu k} \quad \& \quad \varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}$$

(6 marks)

Question three (continued)

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(b) If an earth satellite of $500\ kg$ mass is having a pure tangential speed $v_\theta\left(= r\,\dot{\theta}\right) = 9{,}000\ \frac{m}{s}$ at its near-earth-point $800\ km$ above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(4 marks)**

(ii) calculate the values of the eccentricity ε and show that the orbit is an elliptical orbit. Also calculate its period. **(2+4 marks)**

(iii) what should be the minimum value of the v_θ at the same given near-earth-point such that the satellite would have a open orbit ? **(3 marks)**

Question four

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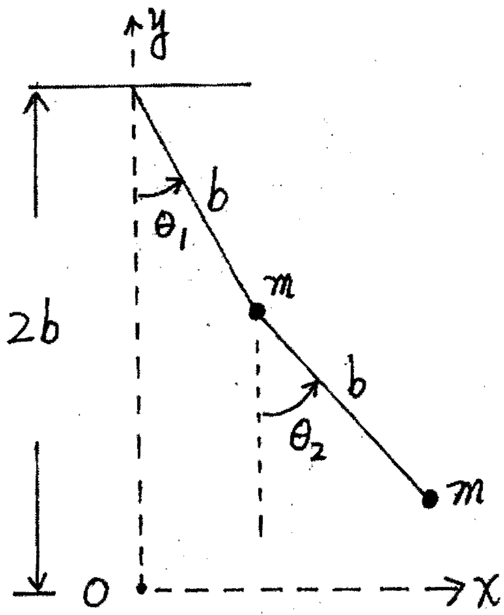
Consider the motion of the bobs in the double pendulum system in the figure below.



Both pendulums are identical and having the length b and bob mass m. The motion of both bobs is restricted to lie in the plane of this paper, i.e., x-y plane.

(a) (i) For small θ_1 *and* θ_2, i.e., $\left(\sin(\theta) \approx \theta \;\; and \;\; \cos(\theta) \approx 1 - \frac{\theta^2}{2} \;\; or \;\; 1\right)$, show that the Lagrangian for the system can be expressed as:

$$L = m\, b^2\, \dot{\theta}_1^2 + \frac{1}{2} m\, b^2\, \dot{\theta}_2^2 + m\, b^2\, \dot{\theta}_1\, \dot{\theta}_2 - m\, g\, b \left(1 + \theta_1^2 + \frac{\theta_2^2}{2}\right) \quad \cdots\cdots \quad (1)$$

where the zero gravitational potential is set at the equilibrium position of the lower bob, i.e., $\theta_1 = 0$, $\theta_2 = 0$ *and* $y = 0$. **(5 marks)**

(ii) Write down the equations of motion and deduce that

$$\begin{cases} 2\,\ddot{\theta}_1 + \ddot{\theta}_2 = -2\frac{g}{b}\,\theta_1 & \cdots\cdots \quad (2) \\ \ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{b}\,\theta_2 & \cdots\cdots \quad (3) \end{cases}$$

(5 marks)

(iii) Deduce from eq.(2) & eq.(3) the following:

$$\begin{cases} \ddot{\theta}_1 = -2\frac{g}{b}\,\theta_1 + \frac{g}{b}\,\theta_2 & \cdots\cdots \quad (4) \\ \ddot{\theta}_2 = 2\frac{g}{b}\,\theta_1 - 2\frac{g}{b}\,\theta_2 & \cdots\cdots \quad (5) \end{cases}$$

(3 marks)

Question four (continued)

(b) (i) Set $\theta_1 = \hat{X}_1 e^{i\omega t}$ *and* $\theta_2 = \hat{X}_2 e^{i\omega t}$ (where $\hat{X}_1$ *and* $\hat{X}_2$ are constants) and deduce from eq.(4) & eq.(5) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(2\frac{g}{b}\right) & \frac{g}{b} \\ 2\frac{g}{b} & -\left(2\frac{g}{b}\right) \end{pmatrix}$$

(3 marks)

(ii) Find the eigenfrequencies ω of this coupled system and show that they are

$$\omega_1 = \sqrt{\left(2-\sqrt{2}\right)\frac{g}{b}} \quad \& \quad \omega_2 = \sqrt{\left(2+\sqrt{2}\right)\frac{g}{b}}$$

(5 marks)

(iii) Find the eigenvector corresponding to ω_1 in (b)(ii). **(4 marks)**

Question five

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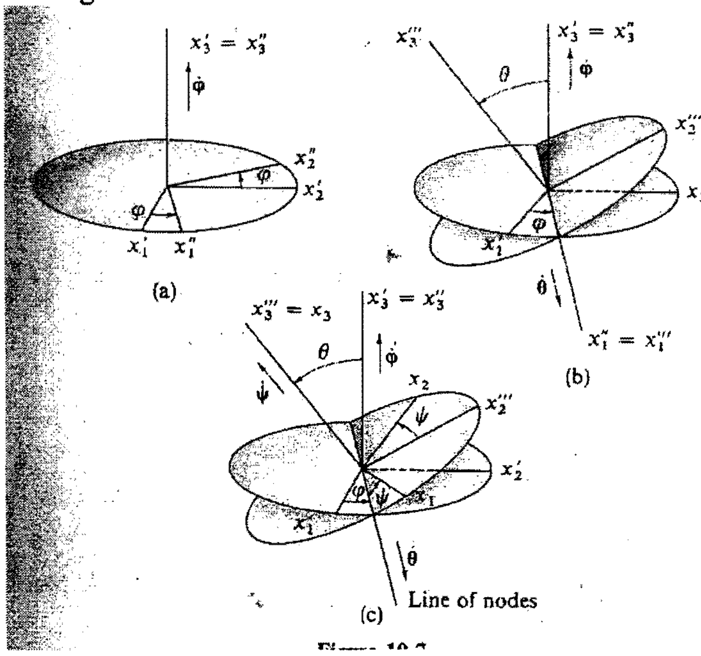
(a) The fixed (or inertia) coordinate system **X'** shares the same origin with the body coordinate system **X** such that only rotational motion is considered. The rotational velocity $\vec{\omega}$ of the body system with respect to the fixed system are breaking down into three independent angular velocities, i.e., $\vec{\omega} = \vec{\dot{\varphi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}$ where (φ, θ, ψ) are Eulerian angles. We use two intermediate coordinate systems **X"** & **X'''** to bridge between **X'** & **X** systems such that $X'' = \lambda_\varphi X'$, $X''' = \lambda_\theta X''$ & $X = \lambda_\psi X'''$ where

$$\lambda_\varphi = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}, \lambda_\psi = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as shown in the figure below.



(i) Since the direction of $\vec{\dot{\varphi}}$ is along x_3'–axis (which is the same as x_3''–axis) with the magnitude of $\dot{\varphi}$ thus $(\vec{\dot{\varphi}})' = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}$ *in X" system* , show that $\vec{\dot{\varphi}}$ in **X** system(i.e., the body system) is $(\vec{\dot{\varphi}}) = \begin{pmatrix} \dot{\varphi}\sin(\theta)\sin(\psi) \\ \dot{\varphi}\sin(\theta)\cos(\psi) \\ \dot{\varphi}\cos(\theta) \end{pmatrix}$ *in X system*. In other words, show that

$$\begin{pmatrix} \dot{\varphi}\sin(\theta)\sin(\psi) \\ \dot{\varphi}\sin(\theta)\cos(\psi) \\ \dot{\varphi}\cos(\theta) \end{pmatrix} = \lambda_\psi \lambda_\theta \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}$$

(5 marks)

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(ii) Since the direction of $\vec{\dot{\theta}}$ is along x_1" –axis (which is the same as x_1''' –axis) with the magnitude of $\dot{\theta}$ thus $\left(\vec{\dot{\theta}}\right)=\begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix}$ *in X''' system*, show that $\vec{\dot{\theta}}$ in X system(i.e., the body system) is $\left(\vec{\dot{\theta}}\right)=\begin{pmatrix}\dot{\theta}\cos(\psi)\\-\dot{\theta}\sin(\psi)\\0\end{pmatrix}$. In other words, show that

$$\begin{pmatrix}\dot{\theta}\cos(\psi)\\-\dot{\theta}\sin(\psi)\\0\end{pmatrix}=\lambda_{\psi}\begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix}$$ **(3 marks)**

(iii) Since any rotational velocity of a rigid body can be expressed as $\vec{\omega}=\vec{\dot{\varphi}}+\vec{\dot{\theta}}+\vec{\dot{\psi}}$ deduce that $\vec{\omega}$ in X system(i.e., body system) in terms of Eulerian angles is

$$(\vec{\omega})\equiv\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix}=\begin{pmatrix}\dot{\varphi}\sin(\theta)\sin(\psi)+\dot{\theta}\cos(\psi)\\\dot{\varphi}\sin(\theta)\cos(\psi)-\dot{\theta}\sin(\psi)\\\dot{\varphi}\cos(\theta)+\dot{\psi}\end{pmatrix}\quad \textit{in X system}$$ **(4 marks)**

(b) (i) By proper choice of the orientation of the body coordinate system, the inertia tensor I (i.e., rotational mass) of a rigid body can be in the form of a diagonalized matrix , i.e., $I=\begin{pmatrix}I_1&0&0\\0&I_2&0\\0&0&I_3\end{pmatrix}$, thus its rotational kinetic is

$$T_{rot}=\frac{1}{2}I_1\,\omega_1^{\ 2}+\frac{1}{2}I_2\,\omega_2^{\ 2}+\frac{1}{2}I_3\,\omega_3^{\ 2}\ .$$

Consider a torque free pure rotational motion of the rigid body, then its Lagrangian is

$$L=T_{rot}=\frac{1}{2}I_1\,\omega_1^{\ 2}+\frac{1}{2}I_2\,\omega_2^{\ 2}+\frac{1}{2}I_3\,\omega_3^{\ 2}\rightarrow L(\varphi,\theta,\psi,\dot{\varphi},\dot{\theta},\dot{\psi})\ .$$

Write down the Lagrange equation of motion for ψ and deduce that

$(I_1-I_2)\,\omega_1\,\omega_2-I_3\,\dot{\omega}_3=0 \quad \cdots\cdots \quad (1)$. **(11 marks)**

(ii) Based on what argument one can write down the other two equations of motion directly from eq.(1) in (b)(i) as $(I_2-I_3)\,\omega_2\,\omega_3-I_1\,\dot{\omega}_1=0$ & $(I_3-I_1)\,\omega_3\,\omega_1-I_2\,\dot{\omega}_2=0$ without going through the similar process of finding the equations of motion for the other two Eulerian angles? **(2 marks)**

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad \text{and reversely} \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \text{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

radius of earth $r_E = 6.4 \times 10^6 \ m$

mass of earth $m_E = 6 \times 10^{24} \ kg$

earth attractive potential $\equiv -\dfrac{k}{r}$ *where* $k = G\, m\, m_E$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k}} \quad \{(\varepsilon = 0, \text{circle}), (0 < \varepsilon < 1, \text{ellipse}), (\varepsilon = 1, \text{parabola}), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_2 >> m_1$$

$$\text{For elliptical orbit, i.e., } 0 < \varepsilon < 1 \text{, then} \begin{cases} \text{semi-major } a = \dfrac{k}{2\,|E|} \\ \text{semi-minor } b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ \text{period } \tau = \dfrac{2\,\mu}{l}(\pi\, a\, b) \\ r_{\min} = a\,(1-\varepsilon) \ \& \ r_{\max} = a\,(1+\varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r\, (\ddot{r} - r\,\dot{\theta}^2) + \vec{e}_\theta\, (2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

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$$I=\begin{pmatrix} \sum_\alpha m_\alpha\left(x_{\alpha,2}^2+x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,2} & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,1} & \sum_\alpha m_\alpha\left(x_{\alpha,1}^2+x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,1} & -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,2} & \sum_\alpha m_\alpha\left(x_{\alpha,1}^2+x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff}=\vec{F}-m\,\ddot{\vec{R}}_f-m\,\dot{\vec{\omega}}\times\vec{r}-m\,\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)-2\,m\,\vec{\omega}\times\vec{v}_r$ *where*

$\vec{r}\,'=\vec{R}+\vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non-inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r=\left(\frac{d\,\vec{r}}{d\,t}\right)_r$$