UNIVERSITY OF SWAZILAND

135

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2013/2014

TITLE OF PAPER : ELECTROMAGNETIC THEORY

COURSE NUMBER : P331

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER <u>ANY FOUR</u> OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES <u>25</u> MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS <u>NINE</u> PAGES, INCLUDING THIS PAGE.

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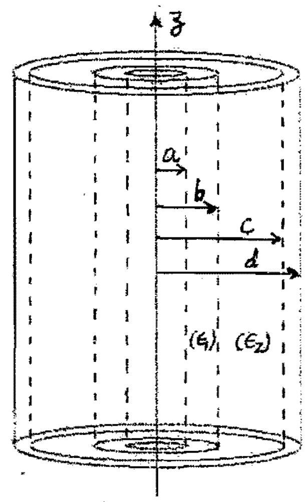
P331 ELECTROMAGNETIC THEORY

136

Question one

A uniform surface charge density of $\left(\dfrac{k}{a}\right)\ C/m^2$ where k is a positive constant is deposited on $\rho = a$ surface of a very long straight coaxial cable, with an inner solid wire of radius $a\ m$. A uniform surface charge density of $\left(-\dfrac{k}{c}\right)\ C/m^2$ is deposited on $\rho = c$ surface of the outer hollow wire with inner radius of $c\ m$ and outer radius of $d\ m$. In-between the wires is filled with two layers of insulating materials with permittivity ε_1 & ε_2 for $(a < \rho < b)$ & $(b < \rho < c)$ layers respectively as shown in the figure below.



(a) Find the total electric charges per unit length deposited on each of the inner and outer conducting wires in terms of k and name them as q_1 & q_2 respectively. Show that $q_1 = -q_2$. **(3 marks)**

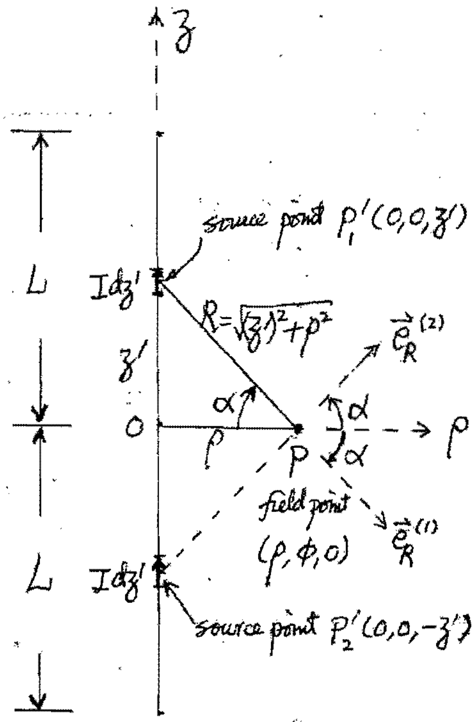
(b) Set $\vec{D}(\rho,\phi,z) = \vec{e}_\rho\ D_\rho(\rho)$ (write a brief justification for this setting), draw an appropriate closed surface and utilize the integral form of electric Gauss law, i.e., $\oint\!\!\oint_S \vec{D} \bullet d\vec{s} = \{total\ Q\ enclosed\ by\ S\}$, to find $D_\rho(\rho)$ for $a < \rho < c$ region. Then write down the form of electric field $\vec{E}$ for $a < \rho < c$ region. **(2 +2 + 6 marks)**

(c) Find the potential difference between the inner and outer wires and then write down the distributive capacitance of the given coaxial cable system. **(5 + 2 marks)**

(d) Explain briefly the bounded charge excess on the surface of a dielectric material and then find the surface charge density of the bounded charge excess σ_{sp} on the $\rho = a$ & $\rho = c$ dielectric material surfaces in terms of k, ε_1 & ε_2. **(1 + 4 marks)**
(Hint : $\sigma_{sp} = \vec{e}_n \bullet \vec{P}$ where $\vec{e}_n$ is the normal outward unit vector on dielectric surface)

Question two

(a) A thin conducting wire of length $2L$, with its central axis coinciding with the z axis and its centre point coinciding with the origin, carries a steady total current I along positive z direction as shown in the figure below.



(i) Since the given current source is only along z axis, its produced vector potential at a field point $P(\rho,\phi,0)$ is also having only z component A_z, i.e.,

$$\vec{A} = \vec{e}_z A_z \quad where \quad A_z = \int_{z'=-L}^{z'=+L} \frac{\mu_0 I dz'}{4\pi R} = 2\int_{z'=0}^{z'=+L} \frac{\mu_0 I dz'}{4\pi\sqrt{(z')^2+\rho^2}},$$

carry out the above integral for A_z about z' and show that

$$A_z = \frac{\mu_0 I}{2\pi}\ln\left(\frac{L+\sqrt{L^2+\rho^2}}{\rho}\right)$$ **(9 marks)**

(Hint : set $z' = \rho\tan(\alpha)$, $\int \sec(\alpha)\,d\alpha = \ln(\sec(\alpha)+\tan(\alpha))$)

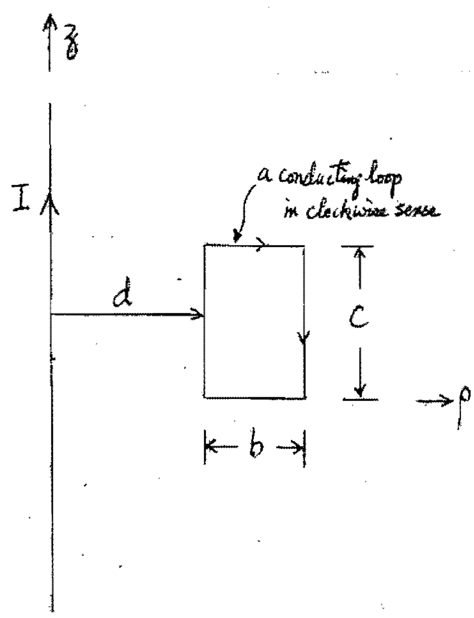
(ii) For $L >> \rho$, i.e., $\sqrt{L^2+\rho^2} \to L$, use $\vec{A} \approx \vec{e}_z \frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{\rho}\right)$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ to find the magnetic field $\vec{B}$ at the field point and show that

$$\vec{B} = \vec{e}_\phi \frac{\mu_0 I}{2\pi\rho}.$$ **(6 marks)**

Question two (continued)

138

(b) Placing a rectangular conducting loop of dimension $b \times c$ a distance of d away from the central current carrying wire as shown in the diagram below, i.e., the inner region confined by the rectangular loop in clockwise sense is
$S:\ d \le \rho \le d+b \ ,\ 0 \le z \le c \ \&\ d\vec{s} = \vec{a}_\phi\, d\rho\, dz$,



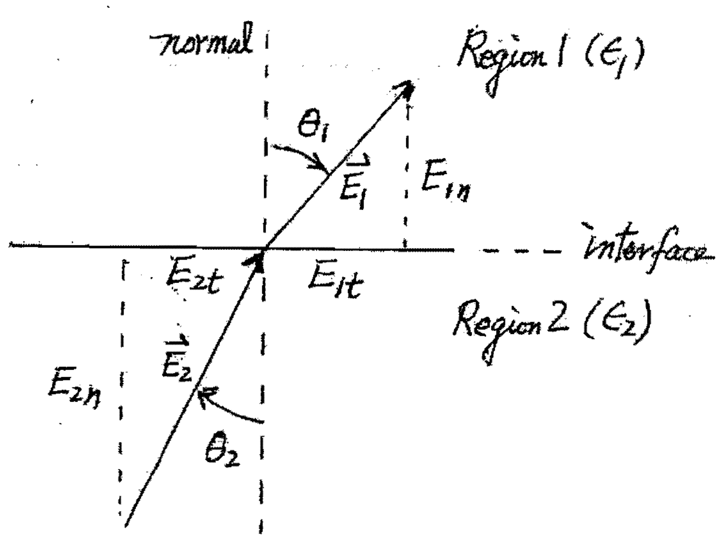
(i) find the total magnetic flux Φ_m passing through the inner region confined by the rectangular loop, i.e., $\Phi_m = \int_S \vec{B} \bullet d\vec{s}$, in terms of $\mu_0, b, c, d \ \&\ I$. Also write down the mutual inductance M between the given rectangular loop and the long straight wire and show that $M = \frac{\mu_0}{2\pi} \times \ln\left(\frac{d+b}{d}\right) \times c$. **(5 marks)**

(ii) If given the values of $b = 5\ cm$, $c = 8\ cm\ \&\ d = 10\ cm$ find the value of M . Further if the wire carries a sinusoidal current $I(t) = 2\sin(9t)\ A$ instead of carrying a static current I , find the induced e.m.f. in the rectangular conducting loop . **(3 + 2 marks)**

Question three

135

(a) An interface separating two dielectric regions of permittivity ε_1 & ε_2 is shown in the figure below. $\vec{E}_1$ & $\vec{E}_2$ are the electric fields at the same point on the interface in the different regions and θ_1 & θ_2 are their respective angles made with the normal.



(i) Use the integral Faraday's law and by choosing proper closed loop across the interface, deduce that the tangential component of $\vec{E}$ is continuous at the interface, i.e., $E_{1t} = E_{2t}$. **(5 marks)**

(ii) Use the integral electric Gauss law and by choosing proper closed surface across the interface, deduce that the normal component of $\vec{D}$ is continuous at the interface, i.e., $D_{1n} = D_{2n}$. **(5 marks)**

(iii) Use the results in (a)(i) and (a)(ii) and deduce the following refraction law for $\vec{E}$ $\varepsilon_1 \tan(\theta_2) = \varepsilon_2 \tan(\theta_1)$. **(4 marks)**

(b) The equation of motion of an average conduction electron based on Drude's model is

$$m_e \frac{d\vec{v}_d}{dt} = -e\vec{E} - \frac{2\, m_e\, \vec{v}_d}{\tau_c} ,$$

(i) Explain briefly each term in the above equation. **(2 marks)**

(ii) In the steady state situation, i.e., $\frac{d\vec{v}_d}{dt} = 0$, deduce the following point form of Ohm's law $\vec{J} = \sigma \vec{E}$ *where* $\sigma = \frac{n e^2}{2\, m_e} \tau_c$, (Hint : $\vec{J} = \rho_v\, \vec{v}_d = -n\, e\, \vec{v}_d$) **(5 marks)**

(iii) If a certain pure metal has an atomic density of $3 \times 10^{28}\ \frac{atoms}{m^3}$ at room temperature and three outer orbit electrons are conduction electrons, find the value of τ_c if its measured dc conductivity is $\sigma = 2 \times 10^7\ \frac{1}{m\,\Omega}$. **(4 marks)**

Question four

140

(a) The Maxwell's equations for the material region of parameters μ , ε & σ are

$$\vec{\nabla} \bullet \vec{E} = 0 \quad \cdots\cdots \quad (1)$$

$$\vec{\nabla} \bullet \vec{H} = 0 \quad \cdots\cdots \quad (2)$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \cdots\cdots \quad (3)$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \cdots\cdots \quad (4)$$

(i) Deduce from them the following wave equation for $\vec{H}$

$$\nabla^2 \vec{H} - \mu \sigma \frac{\partial \vec{H}}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \cdots\cdots \quad (5) \ .$$

(3 marks)

(ii) Set $\vec{H}$ as $\left(\hat{\vec{H}}(space)\, e^{i \omega t}\right)$ and substitute it into the above wave equation, to deduce the following time-harmonic equation for $\hat{\vec{H}}(space)$ as

$$\nabla^2 \hat{\vec{H}}(space) - \hat{\gamma}^2 \hat{\vec{H}}(space) = 0 \qquad where \quad \hat{\gamma} = \sqrt{i \omega \mu \sigma - \omega^2 \mu \varepsilon}$$

(3 marks)

(iii) Set the propagation constant $\hat{\gamma} \equiv \alpha + i \beta$, to deduce that

$$\beta = \frac{\omega \sqrt{\mu \varepsilon}}{\sqrt{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon}\right)^2} + 1} \ .$$

(6 marks)

(Hint : $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1 - \cos(\theta)}{2}}$ & $\cos(\theta) = \left(\sqrt{1 + \tan^2(\theta)}\right)^{-1}$)

(b) An uniform plane wave traveling along the $+z$ direction with the field components $E_x(z)$ & $H_y(z)$ has a complex electric field amplitude $\hat{E}_m = 100\, e^{i\, 20^0} \ \frac{V}{m}$ and propagates at a frequency $f = 10^6 \ Hz$ in a material region having the parameters $\mu = \mu_0$, $\varepsilon = 2\, \varepsilon_0$ & $\frac{\sigma}{\omega \varepsilon} = 0.4$.

(i) Find the values of the propagation constant $\hat{\gamma}$ $(= \alpha + i \beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave , **(4 marks)**

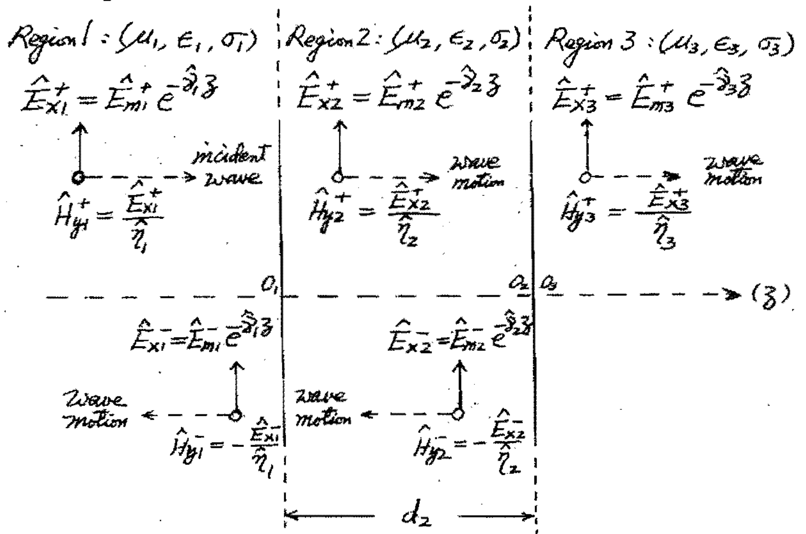
(ii) Express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (b)(i) inserted , **(6 marks)**

(iii) Find the values of the penetration depth, wavelength and phase velocity of the given wave . **(3 marks)**

Question five

141

An uniform plane wave $\left(\hat{E}^{+}_{x1}\ ,\ \hat{H}^{+}_{y1}\right)$, operating at a frequency f , is normally incident upon a layer of d_2 thickness, and emerges to region 3 as shown below :



O_1 , O_2 & O_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface .

(a) Define for the i^{th} *region* $(i = 1,2,3)$ the reflection coefficient $\hat{\Gamma}_i(z)$ and the total wave impedance $\hat{Z}_i(z)$ and deduce the following :

$$\hat{\Gamma}_i(z) = \frac{\hat{Z}_i(z) - \hat{\eta}_i}{\hat{Z}_i(z) - \hat{\eta}_i}$$ **(6 marks)**

(b) If $f = 10^8\ Hz$ & $d_2 = \frac{\lambda_2}{4}$, region 1 & 3 are air regions and region 2 is a lossless region with parameters $\mu_2 = \mu_0$, $\varepsilon_1 = 4\,\varepsilon_0$ & $\frac{\sigma}{\omega\,\varepsilon} = 0$

(i) find the values of β_1 , β_2 , β_3 , λ_2 & $\hat{\eta}_2$. (note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\,\pi\ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(4 marks)**

(ii) Starting with $\hat{\Gamma}_3(z) = 0$ for the rightmost region , i.e., region 3 , and using continuous $\hat{Z}$ at the interface as well as the equations in (a) , find the values of $\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-d_2)$, $\hat{Z}_2(-d_2)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$ **(10 marks)**

(iii) Find the value of $\hat{E}^{-}_{m1}$ & $\hat{E}^{+}_{m2}$ *if given* $\hat{E}^{+}_{m1} = 100\, e^{j\,30^0}\ \frac{V}{m}$ **(5 marks)**

Useful informations

ı42

$$e = 1.6\times10^{-19}\ C$$

$$m_e = 9.1\times10^{-31}\ kg$$

$$\mu_0 = 4\,\pi\times10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85\times10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3\times10^{8}\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\,\pi\quad\Omega = 377\quad\Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B}\bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\oint_L \vec{B}\bullet d\vec{l} = \mu\iint_S \vec{J}\bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E}\bullet d\vec{s}\right)$$

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla}\bullet\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\,\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = \sigma\vec{E}$$

$$\vec{D} = \varepsilon\, \vec{E} = \varepsilon_0\, \vec{E} + \vec{P} \qquad \& \qquad \vec{B} = \mu\, \vec{H} = \mu_0\, \vec{H} + \vec{M}$$

143

$$\oint\!\!\oint_S \vec{F} \bullet d\vec{s} \equiv \oint\!\!\oint\!\!\oint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\, r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\, \rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)}\left(\frac{\partial(F_\phi\, r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\, r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z = \vec{e}_\rho F_\rho + \vec{e}_\phi F_\phi + \vec{e}_z F_z = \vec{e}_r F_r + \vec{e}_\theta F_\theta + \vec{e}_\phi F_\phi$ and

$$d\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r\sin(\theta)\, d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

$$\hat{Z}_i(z) = \hat{\eta}_i \frac{1 + \hat{\Gamma}_i(z)}{1 - \hat{\Gamma}_i(z)} \quad , \quad \hat{\Gamma}_i(z) = \frac{\hat{Z}_i(z) - \hat{\eta}_i}{\hat{Z}_i(z) - \hat{\eta}_i} \quad \&$$

$\hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2\hat{\gamma}_i (z' - z)}$ *where* z' *&* z *are two positions in* i^{th} *region*

$$\vec{D} = \varepsilon\,\vec{E} = \varepsilon_0\,\vec{E} + \vec{P} \qquad \& \qquad \vec{B} = \mu\,\vec{H} = \mu_0\,\vec{H} + \vec{M}$$

$$\oint\!\!\oint_S \vec{F} \bullet d\vec{s} \equiv \oint\!\!\oint\!\!\oint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

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$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2\,\vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\,\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\,r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta\sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x\left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y\left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z\left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\,\rho)}{\partial z}\right) + \vec{e}_\phi\left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\,\rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\,r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\,r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\,r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\,r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x\,F_x + \vec{e}_y\,F_y + \vec{e}_z\,F_z = \vec{e}_\rho\,F_\rho + \vec{e}_\phi\,F_\phi + \vec{e}_z\,F_z = \vec{e}_r\,F_r + \vec{e}_\theta\,F_\theta + \vec{e}_\phi\,F_\phi$ and

$$d\vec{l} = \vec{e}_x\,dx + \vec{e}_y\,dy + \vec{e}_z\,dz = \vec{e}_\rho\,d\rho + \vec{e}_\phi\,\rho\,d\phi + \vec{e}_z\,dz = \vec{e}_r\,dr + \vec{e}_\theta\,r\,d\theta + \vec{e}_\phi\,r\sin(\theta)\,d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

$$\hat{Z}_i(z) = \hat{\eta}_i\,\frac{1 + \hat{\Gamma}_i(z)}{1 - \hat{\Gamma}_i(z)} \quad , \quad \hat{\Gamma}_i(z) = \frac{\hat{Z}_i(z) - \hat{\eta}_i}{\hat{Z}_i(z) - \hat{\eta}_i} \quad \&$$

$\hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\,e^{2\hat{\gamma}_i(z'-z)}$ *where* z' & z *are two positions in* i^{th} *region*