UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2013/2014

TITLE OF PAPER : ELECTROMAGNETIC THEORY

COURSE NUMBER : P331

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS EIGHT PAGES, INCLUDING THIS PAGE.

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P331 ELECTROMAGNETIC THEORY

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Question one

(a) A very long thin conducting wire situated at z-axis and uniformly charged with line charge density ρ_l ,

(i) use integral electric Gauss Law and draw an appropriate Gaussian surface, deduce that the electric field at a field point outside the given thin conducting wire is

$\vec{E} = \vec{e}_\rho \dfrac{\rho_l}{2\,\pi\,\varepsilon_0\,\rho}$ where ρ is the distance from z-axis and

$\vec{e}_\rho$ is one of the unit vectors in cylindrical coordinate system **(2 + 6 marks)**

(ii) use $\Phi = -\int_{P_0}^{P} \vec{E} \bullet d\vec{l}$ to find the electric potential at point P where P_0 is the zero potential reference point here taken as $P_0 : (\rho_0 , 0 , 0)$, deduce that

$$\Phi = \frac{\rho_l}{2\,\pi\,\varepsilon_0} \ln\left(\frac{\rho_0}{\rho}\right)$$ **(4 marks)**

(b) Two long thin conducting wires parallel to z-axis and lying on the $y = 0$ plane , i.e, $x - z$ *plane* , one situated at $x = -\,b$ and carries $-\,\rho_l$ uniform line charge density and the other situated at $x = +\,b$ and carries $+\,\rho_l$ uniform line charge density as shown in the Figure.1 (on $y = 0$ plane) and Figure.2 (on $z = 0$ plane) below :

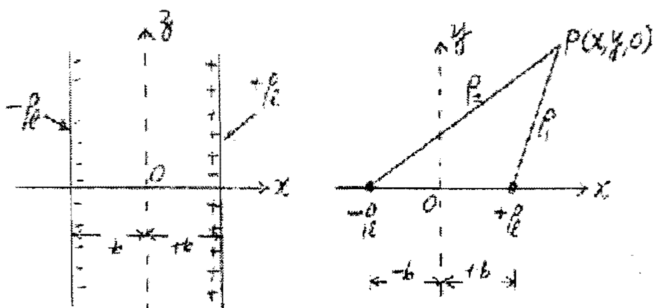

Fig.1 y=0 plane Fig. 2 z=0 plane

(i) Utilize the result in (a)(ii) and choose P_0 as the origin, apply the superposition principle to deduce that the electric potential at point $P : (x , y , 0)$ is

$$\Phi = \frac{\rho_l}{4\,\pi\,\varepsilon_0} \ln\left(\frac{(x+b)^2 + y^2}{(x-b)^2 + y^2}\right)$$ **(6 marks)**

(ii) Set $\dfrac{(x+b)^2 + y^2}{(x-b)^2 + y^2} = K$ where K is a positive constant thus implies Φ is a constant, rewrite it to become a standard form of a circle and show that an equal potential surface is centred at $\left(x_0 = b\,\dfrac{K+1}{K-1} \;,\; y_0 = 0 \right)$ with a radius of $R = \dfrac{2\,\sqrt{K}\,b}{|K-1|}$.

Discuss briefly the limiting cases when $K \to 0^+$ & $K \to +\infty$. **(5 + 2 marks)**

Question two

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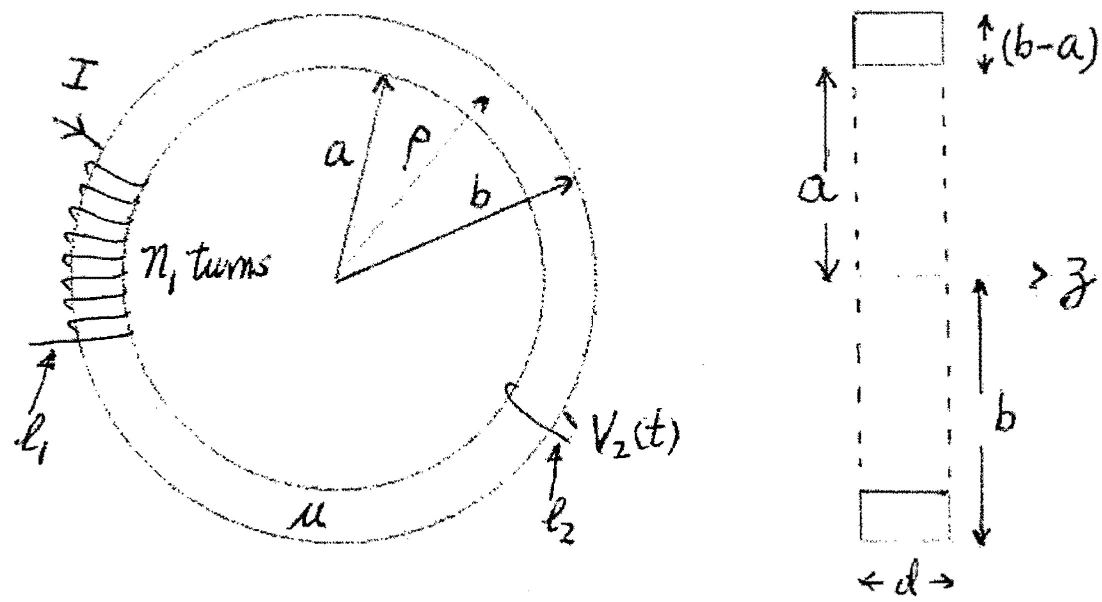
(a) A steady current I flows in the $n_1\ (>>1)$ turns coil l_1 wiring around an iron ring core of magnetic permeability $\mu\ (>>\mu_0)$ with the rectangular cross-section area $(b-a)\times d$ as shown below



(i) Set $\vec{H}(\rho,\phi,z)=\vec{e}_\phi\, H_\phi(\rho)$ (write a brief justification for this setting), draw an appropriate closed loop L and utilize the integral form of Ampere's law, i.e., $\oint_L \vec{H}\bullet d\vec{l} = \{\textit{total I pass through S bounded by L}\}$, to find $H_\phi(\rho)$ for $a<\rho<b$ region. Then write down the magnetic field $\vec{B}$ for $a<\rho<b$ region.

(2 +2 + 5 marks)

(ii) Find the total magnetic flux Ψ_m passing through the cross-section area $(b-a)\times d$ of the iron ring in counter clockwise sense, i.e., $\int_S \vec{B}\bullet d\vec{s}$ where $S: a\le\rho\le b\ ,\ 0\le z\le d\ \&\ d\vec{s}=\vec{a}_\phi\, d\rho\, dz$, in terms of $a, b, d,\ n_1, \mu\ \&\ I$.

(4 marks)

(iii) Find the external self-inductance L_e of l_1 coil and the mutual inductance M between l_1 & l_2 coils where l_2 is a single turn secondary coil. **(5 marks)**

(b) If l_1 coil carries a sinusoidal current of $I_0\sin(\omega t)$ instead of carrying a steady current I , find the induced e.m.f. $V_2(t)$ for a single turn secondary coil l_2 in terms of $a, b, d, \omega,\ n_1\ ,\ \mu\ \&\ I_0$ under quasi static situation . If $a=5\ cm$, $b=6\ cm$, $d=2\ cm$, $n_1=100$, $f=100\ Hz$, $\mu=100\,\mu_0$ and $I_0=3\ A$, compute the amplitude of $V_2(t)$.

(3 + 4 marks)

Question three

(a) The integral form of equation of continuity in Electromagnetic theory is

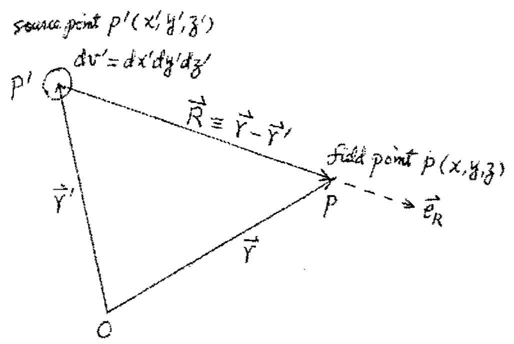
$$\oiint_S \vec{J} \bullet d\vec{s} = -\frac{\partial}{\partial t}\left(\iiint_V \rho_v \, dv \right) \quad \text{where } V \text{ is enclosed by closed surface } S$$

(i) Explain briefly each term of the above equation and indicate which law of Physics it describes. **(3 marks)**

(ii) Utilize the Divergence theory to convert the above integral equation into the differential form of equation of continuity. **(3 marks)**

(iii) Which term in the differential form of Maxwell's equations is the "displacement current" term? Explain in details why Maxwell deemed it necessary to include this "extra" term in Maxwell's equations. **(1 + 6 marks)**

(b) According to Coulomb's law, the electric field $\vec{E}$ at a field point $\vec{r}$ produced by a tiny charge $\rho_v(\vec{r}\,')\,dv'$ at a source point $\vec{r}\,'$ is $\vec{E}(\vec{r}) = \vec{e}_R \dfrac{\rho_v(\vec{r}\,')\,dv'}{4\pi\varepsilon R^2}$ where $\vec{r}$, $\vec{r}\,'$ & $\vec{e}_R$ are shown in the figure below :



From the superposition principle the electric field $\vec{E}$ at a field point $\vec{r}$ produced by all the electric charges is $\vec{E}(\vec{r}) = \iiint\limits_{\forall \vec{r}\,'} \vec{e}_R \dfrac{\rho_v(\vec{r}\,')\,dv'}{4\pi\varepsilon R^2}$ ...... (1)

(i) By direct differentiation, show that $\vec{\nabla}\left(\dfrac{1}{R}\right) = -\vec{e}_R \dfrac{1}{R^2}$ where gradient operator $\vec{\nabla}$ is operating on field point (x , y , z). **(7 marks)**

(*Note* : $\vec{r} = \vec{e}_x\, x + \vec{e}_y\, y + \vec{e}_z\, z$, $\vec{r}\,' = \vec{e}_x\, x' + \vec{e}_y\, y' + \vec{e}_z\, z'$ &

$\vec{R} \equiv \vec{r} - \vec{r}\,' = \vec{e}_x\,(x - x') + \vec{e}_y\,(y - y') + \vec{e}_z\,(z - z') = \vec{e}_R\, R$ *where*

$$\begin{cases} R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \\ \vec{e}_R = \vec{e}_x \dfrac{(x - x')}{R} + \vec{e}_y \dfrac{(y - y')}{R} + \vec{e}_z \dfrac{(z - z')}{R} \end{cases}$$
)

(ii) Take the minus gradient of $f(\vec{r}) = \iiint\limits_{\forall \vec{r}\,'} \dfrac{\rho_v(\vec{r}\,')\,dv'}{4\pi\varepsilon_o R}$ and use the results of (b)(i) and eq.(1) to deduce that $\vec{E}(\vec{r}) \equiv -\vec{\nabla}\left(f(\vec{r})\right)$. **(5 marks)**

Question four

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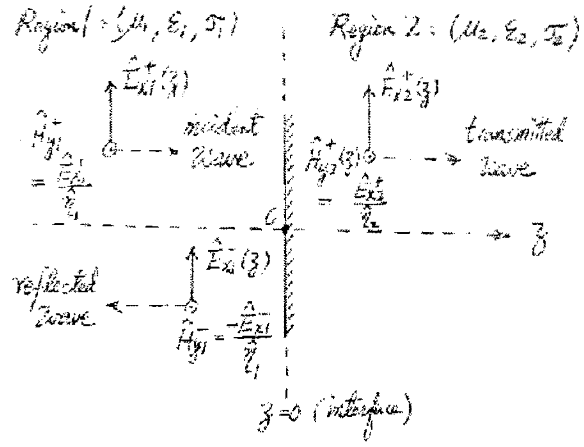
An uniform plane wave is normally incident upon an interface separating two regions as shown below. The incident wave is given as $\left(\hat{E}_{x1}^{+} = \hat{E}_{m1}^{+}\, e^{-\hat{\gamma}_1 z} \,,\, \hat{H}_{y1}^{+} = \dfrac{\hat{E}_{m1}^{+}}{\hat{\eta}_1}\, e^{-\hat{\gamma}_1 z}\right)$ and thus the reflected and transmitted wave can be written as $\left(\hat{E}_{x1}^{-} = \hat{E}_{m1}^{-}\, e^{+\hat{\gamma}_1 z} \,,\, \hat{H}_{y1}^{-} = -\dfrac{\hat{E}_{m1}^{-}}{\hat{\eta}_1}\, e^{+\hat{\gamma}_1 z}\right)$ and $\left(\hat{E}_{x2}^{+} = \hat{E}_{m2}^{+}\, e^{-\hat{\gamma}_2 z} \,,\, \hat{H}_{y2}^{+} = \dfrac{\hat{E}_{m2}^{+}}{\hat{\eta}_2}\, e^{-\hat{\gamma}_2 z}\right)$ respectively as shown in the figure below:



(a) From the boundary conditions at the interface , i.e., both total $\hat{E}_x$ & $\hat{H}_y$ are continuous at z = 0 , deduce the following

$$\begin{cases} \hat{E}_{m1}^{-} = \hat{E}_{m1}^{+}\, \dfrac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \\ \hat{E}_{m2}^{+} = \hat{E}_{m1}^{+}\, \dfrac{2\,\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_1} \end{cases}$$

(11 marks)

(b) If given $\hat{E}_{m1}^{+} = 100\, e^{i\,0^0}\ \dfrac{V}{m}$, $f = 10^6\ Hz$, region 1 is having parameters of $\left(\mu_1 = \mu_0\ ,\ \varepsilon_1 = 4\,\varepsilon_0\ \&\ \dfrac{\sigma_1}{\omega\,\varepsilon_1} = 0.5\right)$ and region 2 is having parameters of $\left(\mu_2 = \mu_0\ ,\ \varepsilon_2 = 9\,\varepsilon_0\ \&\ \dfrac{\sigma_2}{\omega\,\varepsilon_2} = 1\right)$,

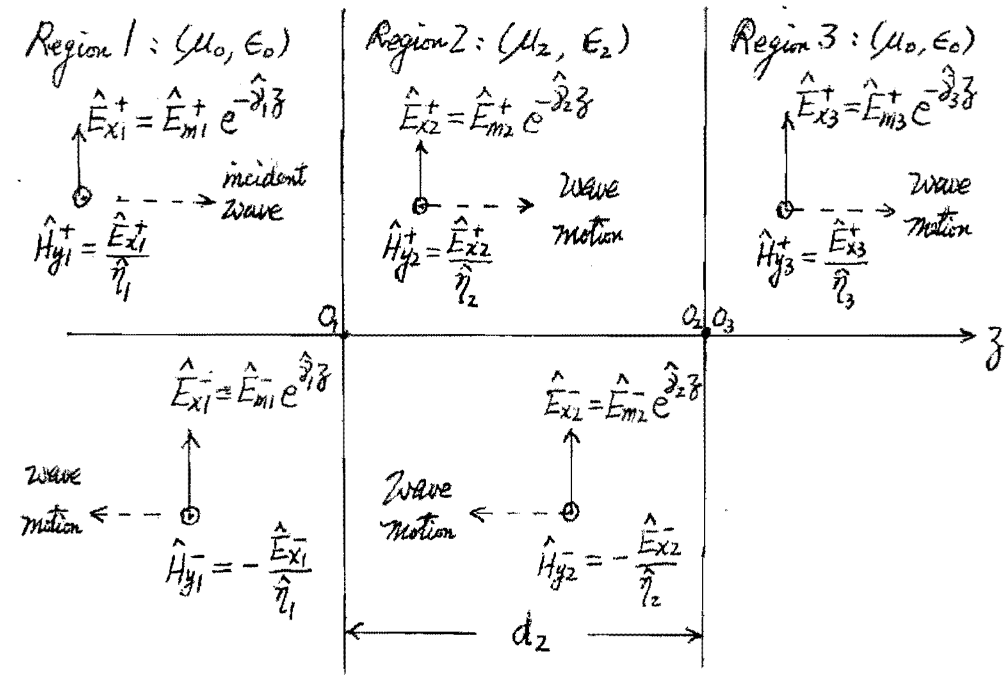
(i) Find the values of $\hat{\eta}_1$ & $\hat{\eta}_2$. **(4 marks)**

(ii) Find the values of $\hat{E}_{m1}^{-}$ & $\hat{E}_{m2}^{+}$ and express them in polar form. **(7 marks)**

(iii) What would be the values of $\hat{E}_{m1}^{-}$ & $\hat{E}_{m2}^{+}$ approaching to if the region 2 is a very good conductive region? **(3 marks)**

Question five

An uniform plane wave $\left(\hat{E}^+_{x1}, \hat{H}^+_{y1}\right)$, operates at $f = 10^7\ Hz$, is normally incident upon a lossless layer of quarter wave thickness $d_2 = \dfrac{\lambda_2}{4}$ with parameters of $\left(\mu_2 = \mu_0,\ \varepsilon_2 = 16\,\varepsilon_0\right)$ as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface . (Both region 1 and region 3 are air region.)

(a) Find the values of β_0 , $\hat{\gamma}_2\ (= \alpha_2 + \beta_2)$ & $\hat{\eta}_2$. **(5 marks)**

(Note : $\hat{\gamma}_1 = \hat{\gamma}_3 = i\,\beta_0 = i\,\omega\sqrt{\mu_0\,\varepsilon_0}$ & $\hat{\eta}_1 = \hat{\eta}_3 = 377 = 120\pi$)

(b) Find all the $\hat{Z}$ & $\hat{\Gamma}$ values at two interfaces , i.e., $\hat{\Gamma}_3(0_3)$, $\hat{Z}_3(0_3)$, $\hat{Z}_2(0_2)$, $\hat{\Gamma}_2(0_2)$, $\hat{\Gamma}_2(-d_2)$, $\hat{Z}_2(-d_2)$, $\hat{Z}_1(0_1)$ & $\hat{\Gamma}_1(0_1)$ **(10 marks)**

(c) Find the values of $\hat{E}^-_{m1}$, $\hat{E}^+_{m2}$, $\hat{E}^-_{m2}$ & $\hat{E}^+_{m3}$ if given $\hat{E}^+_{m1} = 100\,e^{i\,0^0}\ \dfrac{V}{m}$ **(10 marks)**

Useful informations

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$$e = 1.6\times10^{-19}\ C$$

$$m_e = 9.1\times10^{-31}\ kg$$

$$\mu_0 = 4\,\pi\times10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85\times10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3\times10^{8}\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\ e^{\,i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\,\pi\quad\Omega\ = 377\quad\Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\ dv$$

$$\oiint_S \vec{B}\bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\oint_L \vec{B}\bullet d\vec{l} = \mu\iint_S \vec{J}\bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E}\bullet d\vec{s}\right)$$

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla}\bullet\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

$\vec{D}=\varepsilon\,\vec{E}=\varepsilon_0\,\vec{E}+\vec{P}$ & $\vec{B}=\mu\,\vec{H}=\mu_0\,\vec{H}+\vec{M}$ ١٥١

$\oiint_S \vec{F}\bullet d\vec{s}\equiv\iiint_V \left(\vec{\nabla}\bullet\vec{F}\right)dv$ *divergence theorem*

$\oint_L \vec{F}\bullet d\vec{l}\equiv\iint_S \left(\vec{\nabla}\times\vec{F}\right)\bullet d\vec{s}$ *Stokes' theorem*

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{F}\right)\equiv 0$$

$$\vec{\nabla}\times\left(\vec{\nabla}\,f\right)\equiv 0$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right)\equiv\vec{\nabla}\left(\vec{\nabla}\bullet\vec{F}\right)-\nabla^2\,\vec{F}$$

$$\vec{\nabla}\,f=\vec{e}_x\,\frac{\partial f}{\partial x}+\vec{e}_y\,\frac{\partial f}{\partial y}+\vec{e}_z\,\frac{\partial f}{\partial z}=\vec{e}_\rho\,\frac{\partial f}{\partial\rho}+\vec{e}_\phi\,\frac{1}{\rho}\frac{\partial f}{\partial\phi}+\vec{e}_z\,\frac{\partial f}{\partial z}$$

$$=\vec{e}_r\,\frac{\partial f}{\partial r}+\vec{e}_\theta\,\frac{1}{r}\frac{\partial f}{\partial\theta}+\vec{e}_\phi\,\frac{1}{r\sin(\theta)}\frac{\partial f}{\partial\phi}$$

$$\vec{\nabla}\bullet\vec{F}=\frac{\partial(F_x)}{\partial x}+\frac{\partial(F_y)}{\partial y}+\frac{\partial(F_z)}{\partial z}=\frac{1}{\rho}\frac{\partial(F_\rho\,\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial(F_\phi)}{\partial\phi}+\frac{\partial(F_z)}{\partial z}$$

$$=\frac{1}{r^2}\frac{\partial(F_r\,r^2)}{\partial r}+\frac{1}{r\sin(\theta)}\frac{\partial(F_\theta\sin(\theta))}{\partial\theta}+\frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial\phi}$$

$$\vec{\nabla}\times\vec{F}=\vec{e}_x\left(\frac{\partial(F_z)}{\partial y}-\frac{\partial(F_y)}{\partial z}\right)+\vec{e}_y\left(\frac{\partial(F_x)}{\partial z}-\frac{\partial(F_z)}{\partial x}\right)+\vec{e}_z\left(\frac{\partial(F_y)}{\partial x}-\frac{\partial(F_x)}{\partial y}\right)$$

$$=\frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial\phi}-\frac{\partial(F_\phi\,\rho)}{\partial z}\right)+\vec{e}_\phi\left(\frac{\partial(F_\rho)}{\partial z}-\frac{\partial(F_z)}{\partial\rho}\right)+\frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\,\rho)}{\partial\rho}-\frac{\partial(F_\rho)}{\partial\phi}\right)$$

$$=\frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\,r\sin(\theta))}{\partial\theta}-\frac{\partial(F_\theta\,r)}{\partial\phi}\right)+\frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial\phi}-\frac{\partial(F_\phi\,r\sin(\theta))}{\partial r}\right)+\frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\,r)}{\partial r}-\frac{\partial(F_r)}{\partial\theta}\right)$$

where $\vec{F}=\vec{e}_x\,F_x+\vec{e}_y\,F_y+\vec{e}_z\,F_z=\vec{e}_\rho\,F_\rho+\vec{e}_\phi\,F_\phi+\vec{e}_z\,F_z=\vec{e}_r\,F_r+\vec{e}_\theta\,F_\theta+\vec{e}_\phi\,F_\phi$ and

$$d\vec{l}=\vec{e}_x\,dx+\vec{e}_y\,dy+\vec{e}_z\,dz=\vec{e}_\rho\,d\rho+\vec{e}_\phi\,\rho\,d\phi+\vec{e}_z\,dz=\vec{e}_r\,dr+\vec{e}_\theta\,r\,d\theta+\vec{e}_\phi\,r\sin(\theta)\,d\phi$$

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x^2}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}$$

$$\hat{Z}_i(z)=\hat{\eta}_i\,\frac{1+\hat{\Gamma}_i(z)}{1-\hat{\Gamma}_i(z)}\quad,\quad\hat{\Gamma}_i(z)=\frac{\hat{Z}_i(z)-\hat{\eta}_i}{\hat{Z}_i(z)-\hat{\eta}_i}\quad\&$$

$\hat{\Gamma}_i(z')=\hat{\Gamma}_i(z)\,e^{2\hat{\gamma}_i(z'-z)}$ *where* z' & z *are two positions in* i^{th} *region*