UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION: 2013/2014

TITLE OF THE PAPER: QUANTUM MECHANICS

COURSE NUMBER: P342

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

- ANSWER ANY FOUR OUT THE FIVE QUESTIONS.
- EACH QUESTION CARRIES 25 MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN PAGE 2 WHEN NECESSARY.

THIS PAPER HAS 7 PAGES, INCLUDING THIS PAGE.

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Useful Formulas

Time-dependent Schrodinger equation: $i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t)$

Time-independent Schrodinger equation: $\hat{H}\psi(x) = E\psi(x)$

Hamiltonian operator : $\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + V(x)$

Momentum operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Probability current $J(x,t) = \frac{i\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right)$

Fourier transform $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}dk$

Inverse Fourier transform $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx$

$\int_{-\infty}^{\infty} x^{2n}e^{-ax^2}dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}$, n =0, 1, 2, 3, ...

$\int_{-\infty}^{\infty} x^{2n+1}e^{-ax^2}dx = 0$, n =0, 1, 2, 3, ...

Heisenberg uncertainty $\Delta x\Delta p \geq \hbar/2$

Uncertainty of a quantity $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$

Infinite potential well: $V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{elsewhere} \end{cases}$, $E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}$ and $u_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$

Question 1

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(a) Consider a particle confined to the region $0 \leq x \leq L$ with the wavefunction

$$\psi(x,t) = A \sin\left(\frac{2\pi x}{L}\right) e^{-iEt},$$

(i) Find a value for A that normalizes $\psi(x,t)$

[4 marks]

(ii) Determine the eigenvalue energy E in terms of $\hbar$, L and m such that $\psi(x,t)$ satisfies the Schrodinger equation.

[4 marks]

(iii) Sketch the probability density $P(x,t)$ for finding the particle at point x at time t. Label your axes. What is the probability density $P(L/2,t)$ for the particle to be $x = L/2$ at time t.

[4 marks]

(iv) Compute the expectation value for the position $\langle x \rangle$ and the momentum $\langle p \rangle$.

[6 marks]

(b) Consider a particle of mass m confined to a very thin wire of length L; you can treat this as a 1-dimensional problem. Use the uncertainty relation $\Delta p \Delta x \geq \hbar/2$ to estimate the lowest energy that this particle can have, in terms of m, L, and $\hbar$.

[7 marks]

Question 2

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Consider two orthonormal energy eigenstates $|1\rangle$ and $|2\rangle$, where $\hat{H}|1\rangle = E_1|1\rangle$. $\hat{H}|2\rangle = E_2|1\rangle$. $\hat{H}$ being the Hamiltonian and $E_1 \neq E_2$. Let $|A\rangle$ and $|B\rangle$ define two different linear combinations of the states $|1\rangle$ and $|2\rangle$:

$$|A\rangle = \frac{|1\rangle + i|2\rangle}{\sqrt{2}}, \quad |B\rangle = \frac{|1\rangle - i|2\rangle}{\sqrt{2}}$$

(a) Compute $\langle A|A\rangle$, $\langle B|B\rangle$,$\langle A|B\rangle$, and $\langle B|A\rangle$.

[8 marks]

(b) If initially at time $t = 0$ the particle is in the state $|\psi\rangle = |A\rangle$, what is the wavefunction $|\psi, t\rangle$ at later times?

[2 marks]

(c) For the above initial condition $|\psi, 0\rangle = |A\rangle$, what are the probabilities that a measurement at time $t > 0$ will find the particles in the state $|A\rangle$ or in state $|B\rangle$ respectively?

[5 marks]

(d) Suppose at $t = 0$ a particle is in the state

$$|\psi, 0\rangle = N\left(|1\rangle - i|2\rangle + 2|3\rangle + |\sqrt{3}\rangle\right)$$

where $|n\rangle$ are the orthonormal eigenstates of some Hamiltonian with $\hat{H}|n\rangle = E_n|n\rangle$. The number N is chosen to normalize the state so the $\langle\psi, 0|\psi, 0\rangle = 1$. Which of the following statements is *false*?

(i) At later times t, $\langle\psi, t|\psi, t\rangle = 1$.

(ii) The normalization constant can be taken to be $N = 1/3$ (c) the expectation value of the Hamiltonian $\langle\psi, t|\hat{H}|\psi, t\rangle$, is time independent.

(iii) The probability of measuring the energy to be E_3 is twice the probability of measuring the energy to be E_1.

(iv) A measurement of the particle's energy can only yield the values E_1, E_2, E_3 or E_4.

[5 marks] 10

Question 3

(a) A particle in a central potential has an orbital angular momentum $\vec{L}$ and a spin $\vec{S}$. The spin-orbit interaction gives rise to the Hamiltonian $H = A\vec{L} \cdot \vec{S}$ where A is a constant. Let $\vec{J} = \vec{L} + \vec{S}$. Write the Hamiltonian in terms of J^2, L^2, and S^2.

[3 marks]

(b) If L_x, L_y and L_z are the three angular momentum operators in three dimensions. With z-axis as the rotational axis compute the following commutation relations:

(i) $[L_x, L_y]$

(ii) $[L_z, L_y]$

(iii) $[L^2, L_y]$

(iv) $[L^2, L_z]$

(v) $[L_+, L_x]$

(vi) $[L_+, L_z]$

(vii) $[L_+, L_-]$

where $L_\pm = L_x \pm iL_y$.

[7 marks]

(c) For the simple harmonic oscillator ladder operators $\hat{a}$ and $\hat{a}^\dagger$, where $[\hat{a}, \hat{a}^\dagger] = 1$, compute both $[\hat{a}\hat{a}^\dagger, \hat{a}^\dagger]$ and $[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger]$

[5 marks]

(d) Which two quantities describing a particle's motion in a central potential in 3-dimenions *cannot* be simultaneously mesured with arbitrary accuracy, even in principle:

(a) Total angular momentum and the x-component of the angular momentum.

(b) y position and the z-component of angular momentum.

(c) x position and the x-component of the angular momentum.

(d) Energy and total angular momentum.

(e) Energy and the x-component of angular momentum.

[10 marks]

Question 4

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The expectation value of any operator $\hat{O}$ on a state $|\psi, t\rangle$ which is a solution to the time dependent Schrodinger equation satisfies:

$$\frac{d}{dt}\langle\psi, t|\hat{O}|\psi, t\rangle = \frac{i}{\hbar}\langle\psi, t|[\hat{H}, \hat{O}]|\psi, t\rangle$$

(a) Use the commutation relation between the operators x and p to obtain the equations describing the time dependence of $\langle x\rangle$ and $\langle p\rangle$ for the Hamiltonian given by

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m(\omega_1^2 x^2 + \omega_2 x + C)$$

[18 marks]

(b) Solve the equations of motion you obtained in (a). Write your solutions in terms of $\langle x\rangle$ and $\langle p\rangle$, the expectation values at time $t = 0$.

[7 marks]

Question 5

Consider a particle with mass m in a potential

$$V(x) = \begin{cases} \infty \text{ for } x < 0; x > a \\ 0 \qquad 0 < x < a \end{cases}$$

with a initial normalized wave function of the form

$$\psi(x,0) = A\cos^4\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right).$$

(a) What is the form $\psi(x,t)$?

[9 marks]

(b) Calculate the normalization constant A.

[9 marks]

(c) What is the probability that an energy measurement yields E_3, where $E_n = n^2\pi^2\hbar^2/(2ma^2)$?

[8 marks]