UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2015/2016

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) Given the following relations between the unit vectors of cylindrical, spherical and Cartesian coordinate systems as

$$\begin{cases} \vec{e}_\rho = \vec{e}_x \cos(\phi) + \vec{e}_y \sin(\phi) \\ \vec{e}_\phi = -\vec{e}_x \sin(\phi) + \vec{e}_y \cos(\phi) \end{cases} \quad \& \quad \begin{cases} \vec{e}_r = \vec{e}_\rho \sin(\theta) + \vec{e}_z \cos(\theta) \\ \vec{e}_\theta = \vec{e}_\rho \cos(\theta) - \vec{e}_z \sin(\theta) \end{cases},$$

and deduce the following:

(i) $\dfrac{d\vec{e}_\phi}{dt} = -\vec{e}_\rho \dfrac{d\phi}{dt}$ in terms of cylindrical unit vectors ; **(3 marks)**

(ii) $\dfrac{d\vec{e}_\phi}{dt} = -\vec{e}_r \sin(\theta)\dfrac{d\phi}{dt} - \vec{e}_\theta \cos(\theta)\dfrac{d\phi}{dt}$ in terms of spherical unit vectors . **(5 marks)**

(b) Given $\vec{F} = \vec{e}_x\,(2\,x\,y) + \vec{e}_y\,(x^2) + \vec{e}_z\,(-3\,z^2)$ and find the value of

$\int_{P_1,\,L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (1, 2, 0)$, $P_2 : (7, 10, 0)$ and

(i) L : a straight line from P_1 to P_2 on $x-y$ plane , i.e., $z = 0$ plane ; **(6 marks)**

(ii) L : a parabolic path $y = \dfrac{1}{6}x^2 + \dfrac{11}{6}$ from P_1 to P_2 on $x-y$ plane.
Compare this answer with that obtained in (b)(i) and comment on the conservative nature of the given vector field. **(6 + 1 marks)**

(iii) Find $\vec{\nabla} \times \vec{F}$. Does this answer in agreement with the comment in (b)(ii) ? **(3 + 1 marks)**

Question two

(a) Given a scalar function in cylindrical coordinates as $f = \rho^2 \cos(\phi) - 4\, z^2$,

(i) find the value of $\vec{\nabla} f$ at a point $P : (\,10\,, 240^0\,, -2\,)$, **(3 marks)**

(ii) find the value of the directional derivative of f at a point $P : (10\,, 240^0\,, -2)$ along the direction of $\vec{e}_\rho\, 8 - \vec{e}_\phi\, 4 + \vec{e}_z$, **(3 marks)**

(iii) find $\vec{\nabla} \times \left(\vec{\nabla} f\right)$ and shows that it is zero. **(3 marks)**

(b) Given a vector field $\vec{F} = \vec{e}_r\,(r\cos(\theta)) + \vec{e}_\theta\,(-r) + \vec{e}_\phi\,(3\, r \sin\phi)$ in spherical coordinates ,

(i) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if $S = S_1 + S_2$ where

$$S_1 \; : \left(\begin{array}{l} r = 3 \;,\; 0 \le \theta \le \dfrac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = \vec{e}_r\, r^2 \sin\theta\, d\theta\, d\phi \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{r=3} \vec{e}_r\; 9 \sin\theta\, d\theta\, d\phi \end{array} \right)$$

$$S_2 \; : \left(\begin{array}{l} \theta = \dfrac{\pi}{2} \;,\; 0 \le r \le 3 \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = \vec{e}_\theta\, r \sin\theta\, dr\, d\phi \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\theta = \frac{\pi}{2}} \vec{e}_\theta\; r\, dr\, d\phi \end{array} \right)$$

i.e., S is a upper-half semi-spherical closed surface centered at the origin with a radius of 3 , **(8 marks)**

(ii) find $\vec{\nabla} \bullet \vec{F}$ and then evaluate the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in (b)(i) , i.e.,

$$V : \; 0 \le r \le 3 \;,\; 0 \le \theta \le \frac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \quad \& \quad dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \;.$$

Compare this answer to that obtained in (b)(i) and make a brief comment.

(7 + 1 marks)

Question three

Given the following non-homogeneous differential equation as

$$\frac{d^2 y(t)}{dt^2} + 6\frac{d\,y(t)}{dt} + 25\,y(t) = f(t) \quad \cdots\cdots \quad (1)$$

(a) (i) if $f(t) = 102\cos(t) + 195\sin(2t)$ in eq.(1), find its particular solution $y_p(t)$ and show that

$$y_p(t) = 4\cos(t) + \sin(t) - 4\cos(2t) + 7\sin(2t) \quad \cdots\cdots \quad (2),$$ **(7 marks)**

(ii) if $f(t) = 50\,t^2 + 24\,t$ in eq.(1), find its particular solution and show that

$$y_p(t) = 2\,t^2 - \frac{4}{25} \quad \cdots\cdots \quad (3)$$ **(5 marks)**

(iii) explain why $y_p(t)$ is called the steady state solution of the given non-homogeneous differential equation no matter what the given initial conditions are. **(2 mark)**

(b) The homogeneous part of the given equation is $\frac{d^2 y(t)}{dt^2} + 6\frac{d\,y(t)}{dt} + 25\,y(t) = 0$.

Find its general solution $y_h(t)$ and show that $y_h(t) = k_1\,e^{-3t}\cos(4t) + k_2\,e^{-3t}\sin(4t)$ where k_1 & k_2 are arbitrary constants. **(4 marks)**

(c) If $f(t) = 102\cos(t) + 195\sin(2t)$ in eq.(1), the general solution to the given non-homogeneous differential equation $y_g(t)$ can be written as

$$y_g(t) = y_h(t) + y_p(t)$$
$$= \left(k_1\,e^{-3t}\cos(4t) + k_2\,e^{-3t}\sin(4t)\right) + \left(4\cos(t) + \sin(t) - 4\cos(2t) + 7\sin(2t)\right) \quad \cdots \quad (4)$$

Find its specific solution $y_s(t)$ if the initial conditions are given as

$$y(0) = +3 \quad \& \quad \left.\frac{d\,y(t)}{dt}\right|_{t=0} = -1$$, and show that

$$y_s(t) = \left(3\,e^{-3t}\cos(4t) - \frac{7}{4}e^{-3t}\sin(4t)\right) + \left(4\cos(t) + \sin(t) - 4\cos(2t) + 7\sin(2t)\right)$$

(7 marks)

Question four

The longitudinal vibration amplitude $u(x,t)$ of a given vibrating string of length 10 meters, fixed at its two ends, i.e., $u(0,t)=0$ & $u(10,t)=0$, and satisfies the following 1-D wave equation $\frac{\partial^2 u(x,t)}{\partial t^2} = 25\,\frac{\partial^2 u(x,t)}{\partial x^2}$ ...... (1)

(a) set $u(x,t)=F(x)\,G(t)$ and apply the techniques of separation of variables to deduce the following two ordinary differential equations that

$$\begin{cases} \dfrac{d^2 F(x)}{dx^2} = \dfrac{k}{25}\,F(x) & \cdots\cdots \quad (2) \\[2ex] \dfrac{d^2 G(t)}{dt^2} = k\,G(t) & \cdots\cdots \quad (3) \end{cases}$$

where k is a separation constant.

For our given problem, k needs to be any negative constant, explain briefly why ?

(4 + 2 marks)

(b) Consider the following $u(x,t)$ that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \qquad where \quad u_n(x,t) = E_n \sin\left(\frac{n\pi x}{10}\right)\cos\left(\frac{n\pi t}{2}\right) ,$$

(i) by direct substitution, show that $u_n(x,t)$ satisfies the given 1-D wave equation eq.(1), **(4 marks)**

(ii) show that $u_n(x,t)$ satisfies the two fixed conditions, i.e., $u_n(0,t)=0$ & $u_n(10,t)=0$, **(2 marks)**

(iii) show that $u_n(x,t)$ satisfies the zero initial speed condition, i.e.,

$$\left.\frac{\partial u_n(x,t)}{\partial t}\right|_{t=0} = 0 ,$$ **(3 marks)**

(iv) if the initial position of the vibrating string, i.e., $u(x,0)$, is given as

$$u(x,0) = \begin{cases} 2x & for \quad 0 \le x \le 6 \\ -3x+30 & for \quad 6 \le x \le 10 \end{cases} ,$$

find the values of E_n and show that

$$E_n = \frac{100}{n^2 \pi^2} \sin\left(\frac{3n\pi}{5}\right) \quad where \quad n = 1,2,3,\cdots$$

Also calculate the value of E_1 . **(9 + 1 marks)**

Question five

(a) Given the following differential equation as $\frac{d\,y(x)}{d\,x} + 2\,y(x) = 0 \quad \cdots\cdots \quad (1)$,

(i) by direct substitution , show that e^{-2x} is its independent solution ; **(1 mark)**

(ii) set $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ & $a_0 \neq 0$ and use power series method to find its independent solution truncated up to a_3 terms , then show that this power series solution is linearly dependent to

$$e^{-2x}\left(= 1 - \frac{2}{1!}x + \frac{4}{2!}x^2 - \frac{8}{3!}x^3 + \cdots\cdots \quad \textit{in Taylor series} \right) .$$

(10+1 marks)

(b) Given the following differential equations for a coupled oscillator system as

$$\begin{cases} \frac{d^2 x_1(t)}{d t^2} = -5\, x_1(t) + 3\, x_2(t) \\ \frac{d^2 x_2(t)}{d t^2} = 2\, x_1(t) - 10\, x_2(t) \end{cases}$$

(i) set $x_1(t) = X_1\, e^{i\omega t}$ *and* $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\,X = -\omega^2\, X$ where

$$A = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix} \quad and \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} ,$$ **(3 marks)**

(ii) find the eigenfrequencies ω , **(4 marks)**

(iii) find the eigenvectors corresponding to each eigenfrequencies found in (b)(ii), **(4 marks)**

(iv) write down the general solutions for $x_1(t)$ & $x_2(t)$ in terms of eigenfrequencies and eigenvectors found in (b)(ii) and (b)(iii). **(2 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial (F_1 h_2 h_3)}{\partial u_1} + \frac{\partial (F_2 h_1 h_3)}{\partial u_2} + \frac{\partial (F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial (F_3 h_3)}{\partial u_2} - \frac{\partial (F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\partial (F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial (F_2 h_2)}{\partial u_1} - \frac{\partial (F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$f(t) = f(t+2L) = f(t+4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \textit{where}$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_0 = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \quad \textit{for } n = 1,$$

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$