UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2015/2016

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS SEVEN PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

Given a vector field $\vec{F} = \vec{e}_x\left(x^2\right) + \vec{e}_y\left(y^2 - x\,z\right) + \vec{e}_z\left(x^2\right)$ in Cartesian coordinates,

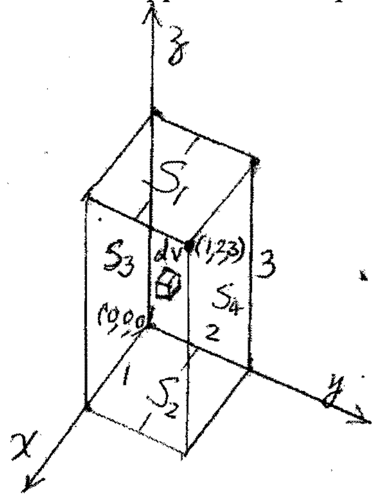
(a) find the value of $\int_{P_1,L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (0,0,1)$, $P_2 : (2,8,1)$ and

(i) L : a straight line from P_1 to P_2 on $z = 1$ plane, **(6 marks)**

(ii) L : a cubic curve $y = x^3$ from P_1 to P_2 on $z = 1$ plane.
Then compare this answer with that obtained in *(a)(i)* and comment on whether the given $\vec{F}$ is a conservative vector field or not. **(6+1 marks)**

(iii) Find $\vec{\nabla} \times \vec{F}$. Does it agree with your comment in *(a)(ii)* ? **(3+1 marks)**

(b) Choose a closed surface S as the cover surface of a rectangular box of dimension $1 \times 2 \times 3$ with its two opposite vertex points as *(0,0,0) and (1,2,3)* as show in the diagram below :



Utilize the Divergence theorem to find the value of $\oint_S \vec{F} \bullet d\vec{s}$, i.e., instead of doing the closed surface integral to find the answer, you can do the volume integral about the volume V which is bounded by the chosen closed surface S to find the same answer. Here doing the volume integral is much easier than that doing the closed surface integral.

($S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ where

$S_1 \;:\; z = 3\,,\, 0 \le x \le 1\,,\, 0 \le y \le 2 \;\;\&\;\; d\vec{s} = +\vec{e}_z\,(dx)(dy)$

$S_2 \;:\; z = 0\,,\, 0 \le x \le 1\,,\, 0 \le y \le 2 \;\;\&\;\; d\vec{s} = -\vec{e}_z\,(dx)(dy)$

$S_3 \;:\; y = 0\,,\, 0 \le x \le 1\,,\, 0 \le z \le 3 \;\;\&\;\; d\vec{s} = -\vec{e}_y\,(dx)(dz)$

$S_4 \;:\; y = 2\,,\, 0 \le x \le 1\,,\, 0 \le z \le 3 \;\;\&\;\; d\vec{s} = +\vec{e}_y\,(dx)(dz)$

$S_5 \;:\; x = 1\,,\, 0 \le y \le 2\,,\, 0 \le z \le 3 \;\;\&\;\; d\vec{s} = +\vec{e}_x\,(dy)(dz)$

$S_6 \;:\; x = 0\,,\, 0 \le y \le 2\,,\, 0 \le z \le 3 \;\;\&\;\; d\vec{s} = -\vec{e}_x\,(dy)(dz)$

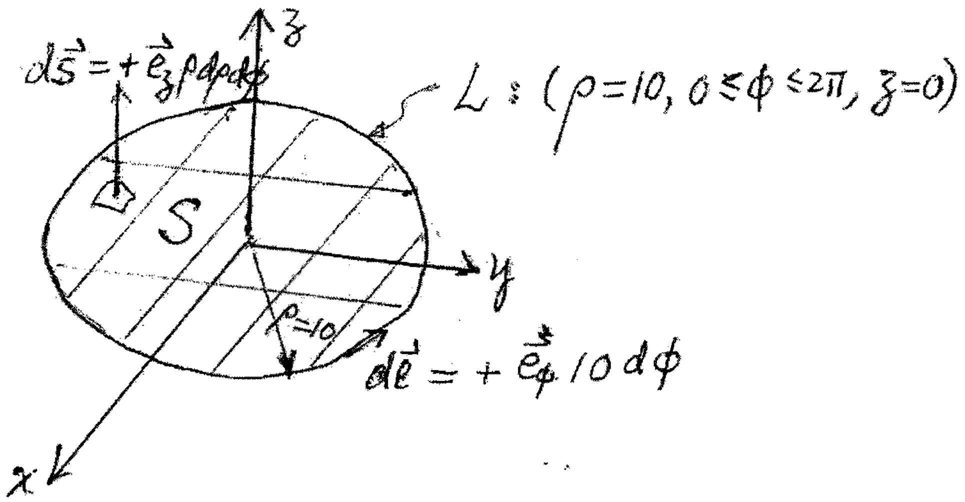
$V : \left(0 \le x \le 1\,,\, 0 \le y \le 2\,,\, 0 \le y \le 2 \;\;\&\;\; dv = (dx)(dy)(dz)\right)$

(8 marks)

Question two

Given $\vec{F} = \vec{e}_\rho \left(z^2 \cos(\phi)\right) + \vec{e}_\phi \left(\rho^2\right) + \vec{e}_z \left(\rho^2 \sin(\phi)\right)$ in cylindrical coordinates,

(a) find the value of $\oint_L \vec{F} \bullet d\vec{l}$ if L is the circular closed loop of radius 10 on $z = 0$ *plane* in counter clockwise sense as shown in the diagram below



i.e., $L : \left(\rho = 10 \ , \ 0 \le \phi \le 2\pi \ , \ \theta = \frac{\pi}{2} \quad \& \quad d\vec{l} = +\vec{e}_\phi \, \rho \, d\phi \xrightarrow{\rho = 10} \vec{e}_\phi \, 10 \, d\phi \right)$

(6 marks)

(b) (i) Find $\vec{\nabla} \times \vec{F}$ and show that

$$\vec{\nabla} \times \vec{F} = \vec{e}_\rho \left(\rho \cos(\phi)\right) + \vec{e}_\phi \left(2\, z \cos(\phi) - 2\, \rho \sin(\phi)\right) + \vec{e}_z \left(3\, \rho^2 + \frac{z^2 \sin(\phi)}{\rho}\right)$$

(5 marks)

(ii) Evaluate the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by L given in (a), i.e., $S : \left(0 \le \rho \le 10 \ , \ 0 \le \phi \le 2\pi \ , \ z = 0 \quad \& \quad d\vec{s} = +\vec{e}_z \, \rho \, d\rho \, d\phi \right)$.
Compare this value with that obtained in (a) and make a brief comment.

(8+1 marks)

(c) Find $\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right)$ and show that it is zero. **(5 marks)**

Question three

Given the following Legendre differential equation as :

$$\left(1-x^2\right)\frac{d^2\,y(x)}{d\,x^2}-2x\,\frac{d\,y(x)}{d\,x}+20\,y(x)=0 \quad \cdots\cdots \quad (1)$$

(a) utilize the power series method, i.e., setting $y(x)=\sum_{n=0}^{\infty} a_n\,x^{n+s}$ *and* $a_0 \neq 0$,

(i) write down the indicial equations. Deduce that $s=0$ *or* 1 and $a_1=0$.

(10 marks)

(ii) Write down the recurrence relation. For $s=0$ case with $a_1=0$, set $a_0=1$ and use the recurrence relation to calculate the values of a_n up to the value of a_8 . Without further calculations, what would be the values of the rest of a_n *for* $n>8$ and give a brief explanation. Thus write down this independent solution in its power series form for all n .

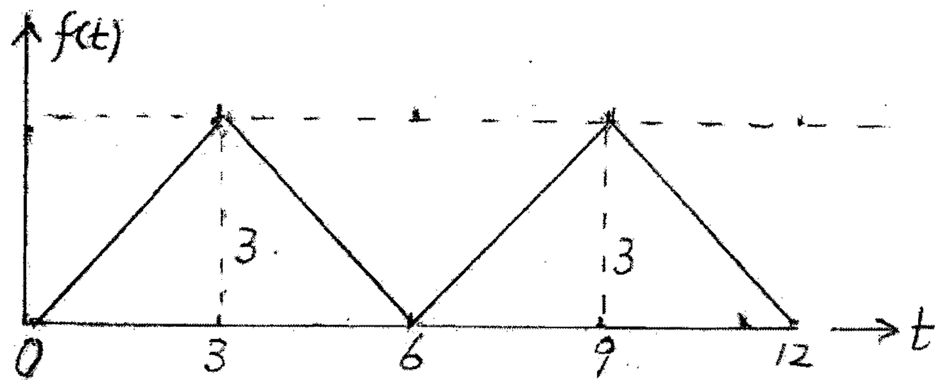
(1+10+2 marks)

(b) The given differential equation in *(a)* has a well-known polynomial solution $P_4(x)$ which is $P_4(x)=\frac{3}{8}+\frac{35}{8}x^4-\frac{15}{4}x^2$. Show that $P_4(x)$ is linearly dependent to the solution you obtained in *(a)(ii)*.

(2 marks)

Question four

Given the following periodic function *f(t)* of period 6 and plotted for two periods from $t = 0$ to $t = 12$ as below :



i.e., for one period of $t = 0$ to $t = 6$, *f(t)* can be wholly described as :

$$f(t) = \begin{cases} t & \text{if} \quad 0 < t < 3 \\ -t + 6 & \text{if} \quad 3 < t < 6 \end{cases}$$

(a) Its Fourier series representation is $f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{3}\right)$.

(i) What special property of the given *f(t)* implies that all the Fourier sine series coefficients should be zero, i.e., $b_n = 0 \quad \forall \; n$. **(2 mark)**

(ii) Find all the Fourier cosine series coefficients and show that

$$a_0 = \frac{3}{2} \quad \&$$

$$a_n = \frac{6\left(\cos(n\pi) - 1\right)}{n^2 \pi^2} \qquad n = 1, 2, 3, \cdots\cdots$$

Thus the Fourier series representation of the given *f(t)* is

$$f(t) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{6\left(\cos(n\pi) - 1\right)}{n^2 \pi^2} \cos\left(\frac{n\pi t}{3}\right) \quad \cdots\cdots \quad (1)$$

(9 marks)

(b) If the above given *f(t)* is the non-homogeneous term for the following non-homogeneous differential equation $\frac{d^2 y(t)}{dt^2} - \frac{d\,y(t)}{dt} + y(t) = f(t)$, find the particular solution $y_p(t)$ to the given periodical *f(t)* represented by its Fourier series in (a)(ii), i.e., *eq.(1)*, and show that

$$y_p(t) = \frac{3}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi t}{3}\right) + B_n \sin\left(\frac{n\pi t}{3}\right) \right) \qquad \textit{where}$$

$$\begin{cases} A_n = -\dfrac{9\left(n^2\pi^2 - 9\right)}{\left(n^4\pi^4 - 9\,n^2\pi^2 + 81\right)} \; a_n \\ B_n = -\dfrac{27\,n\pi}{\left(n^4\pi^4 - 9\,n^2\pi^2 + 81\right)} \; a_n \end{cases} \qquad n = 1, 2, 3, \cdots\cdots$$

(14 marks)

Question five

Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -10\, x_1(t) + 6\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 2\, x_1(t) - 11\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -10 & 6 \\ 2 & -11 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(4 marks)**

(b) Find the eigenfrequencies ω of the given coupled system and show that they are $\omega_1 = \sqrt{7}$ & $\omega_2 = \sqrt{14}$, **(5 marks)**

(c) Find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in *(b)*, **(6 marks)**

(d) Find the normal coordinates $x_1'(t)$ & $x_2'(t)$ of the given coupled system and show that one of the possible answers is

$$\begin{cases} x_1'(t) = 2\, x_1(t) + 3\, x_2(t) \\ x_2'(t) = x_1(t) - 2\, x_2(t) \end{cases}$$

(Note : Normal coordinates just like eigenvectors, they are not unique. Any constant times the answer can be another possible answer.) **(8 marks)**

(e) Write down the general solution of the given system in terms of the eigenfrequencies and eigenvectors obtained in *(b) & (c)*. **(2 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$f(t) = f(t + 2L) = f(t + 4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \textit{where}$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_0 = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \quad \textit{for } n = 1,2$$

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$