UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2015/2016

TITLE OF PAPER	:	**CLASSICAL MECHANICS**
COURSE NUMBER	:	**P320**
TIME ALLOWED	:	**THREE HOURS**
INSTRUCTIONS	:	**ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

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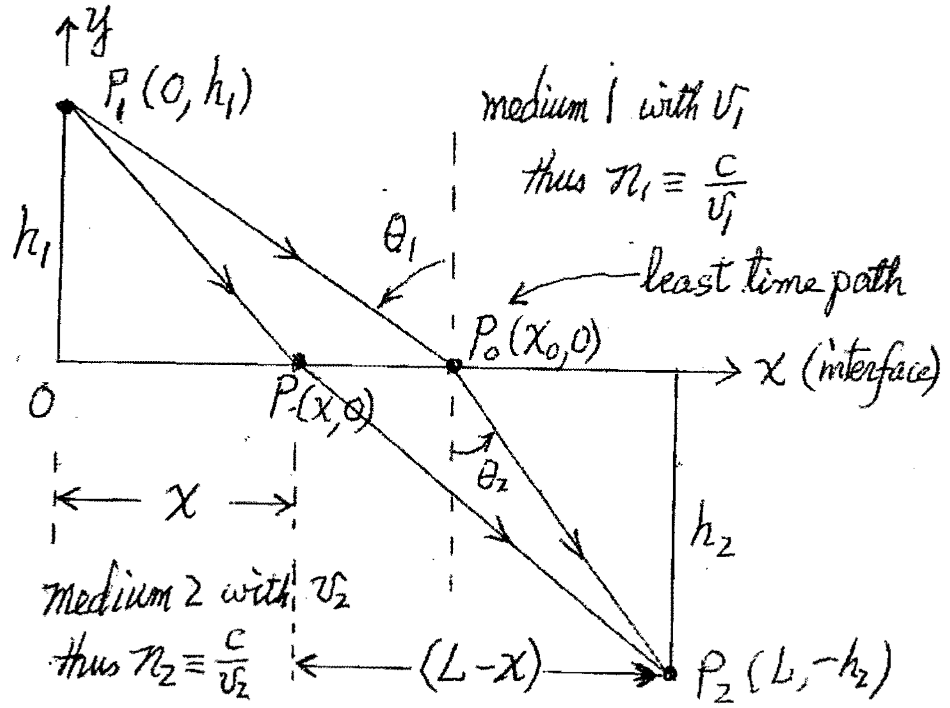
P320 CLASSICAL MECHANICS

Question one

(a) A single frequency light ray travels through $P_1:(0,h_1)$ in medium one with velocity v_1 to reach $P_2:(L,-h_2)$ in medium two with velocity v_2 as shown in the diagram below :



If one of the possible ray passage from $P_1:(0,h_1)$ to $P_2:(L,-h_2)$ is through $P:(x,0)$ *with* $0 \le x \le L$ on the interface $y=0$, then the total time $T(x)$ of this light ray travel from $P_1 \rightarrow P(x,0) \rightarrow P_2$ can be written as

$$T(x)=\frac{\sqrt{x^2+h_1^{\,2}}}{v_1}+\frac{\sqrt{(L-x)^2+h_2^{\,2}}}{v_2} \quad \cdots\cdots \quad (1)$$

(i) Use the least time principle, i.e., $\delta\, T(x)=0$, to find the least time ray passage $P_1 \rightarrow P_0(x_0,0) \rightarrow P_2$ and show that x_0 satisfies

$$\frac{x_0}{v_1\sqrt{x_0^{\,2}+h_1^{\,2}}}=\frac{L-x_0}{v_2\sqrt{(L-x_0)^2+h_2^{\,2}}} \quad \cdots\cdots \quad (2)$$

(4 marks)

(ii) Show that the eq.(2) can be simplified to be the following well-known law of refraction that $n_1 \sin(\theta_1)=n_2 \sin(\theta_2)$ where n_1 & n_2 are the index of refraction for medium one and two respectively. **(3 marks)**

(b) For a particle of mass m acted on by an earth gravitational force of $\vec{F}=-\,\vec{e}_y\, m\, g$ and undergoes a projectile motion near the earth surface in a x-y plane where along the horizontal x direction there is no force acting on the particle .

Question one (continued)

(i) Write down the Hamiltonian H of the system, i.e., $H(x, y, p_x, p_y)$, and show that $H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + m g y$. **(4 marks)**

(ii) From the definition of the Poisson brackets, i.e.,

$$[F,G] \equiv \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right) ,$$

evaluate $[x,H]$, $[y,H]$, $[p_x,H]$ *and* $[p_y,H]$. **(4 marks)**

(iii) For an equation of the type $\frac{du}{dt} = [u,H]$ the specific solution of $u(t)$ is given by the following series expansion

$$u(t) = u_0 + [u,H]_0\, t + [[u,H],H]_0 \frac{t^2}{2!} + [[[u,H],H],H]_0 \frac{t^3}{3!} + \cdots\cdots$$

where subscript 0 denotes the initial conditions at $t = 0$.

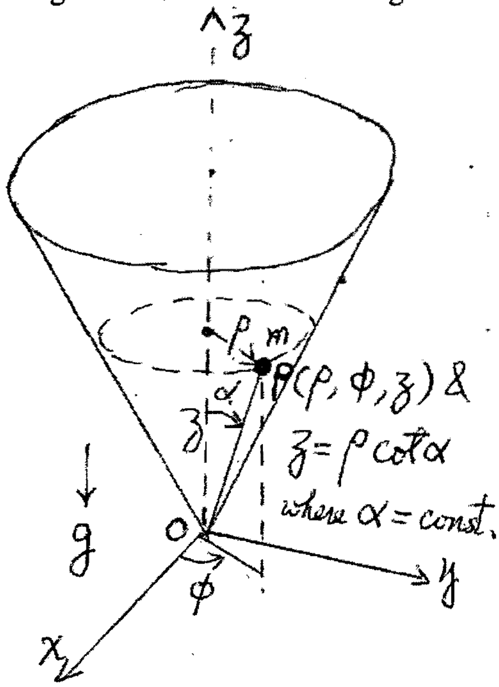
Use the above relation to show that for the given Hamiltonian, the specific solution of $x(t)$ and $y(t)$ are given by

$$\begin{cases} x(t) = x_0 + \frac{p_{x,0}}{m} t \\ y(t) = y_0 + \frac{p_{y,0}}{m} t - \frac{g}{2} t^2 \end{cases}$$

where x_0 *and* $p_{x,0}$ are the initial x-position and x-momentum and y_0 *and* $p_{y,0}$ are the initial y-position and y-momentum . **(10 marks)**

Question two

A particle of mass m is constrained to slide freely on the inner surface of a stationary cone of half-angle α as shown in the diagram below.



The origin O is chosen to be at the bottom of the cone. Assuming the bowl surface is frictionless and the only explicit force acting on the particle is gravitational force with its acceleration g along $-z$ direction.

(a) (i) Write down the Lagrangian for the particle in terms of ρ & ϕ and show that

$$L = \frac{1}{2} m \left(\csc^2(\alpha)\, \dot{\rho}^2 + \rho^2\, \dot{\phi}^2\right) - m\, g\, \rho \cot(\alpha) \quad .$$ **(5 marks)**

(Hint : $\vec{v} \equiv \dfrac{d\vec{s}}{dt}$ and for cylindrical coordinate system one has $z = \rho \cot(\alpha)$ for the cone surface and $(d\vec{s}) \bullet (d\vec{s}) = (ds)^2 = (d\rho)^2 + \rho^2\,(d\phi)^2 + (dz)^2$)

(ii) Write down their respective equations of motion for ρ & ϕ . **(5 marks)**

(iii) Write down their canonical momenta p_ρ & p_ϕ and show that p_ϕ is a constant. **(2 marks)**

(b) (i) Here we can use $H = T + V$ short cut instead of the definition

$H = \sum_{\forall i} (p_i\, \dot{q}_i) - L$ to write down the Hamiltonian of the system (explain briefly why so) and deduce that

$$H = \frac{\sin^2(\alpha)\, p_\rho^{\;2}}{2\,m} + \frac{p_\phi^{\;2}}{2\,m\,\rho^2} + m\, g\, \rho \cot(\alpha)$$ **(1+6 marks)**

(ii) From the Hamiltonian in (b)(i), write down the equations of motion of the system. For each equation obtained here, point out its equivalent equation obtained in (a). **(4+2 marks)**

Question three

(a) The orbital equation of the two-body central force with potential $V(r) = -\frac{k}{r}$ is given as

$$\frac{\alpha}{r} = 1 + \varepsilon \cos(\theta) \qquad where \quad \alpha = \frac{l^2}{\mu k} \quad \& \quad \varepsilon = \sqrt{1 + \frac{2 E l^2}{\mu k^2}} \quad .$$

(i) Set $r = \sqrt{x^2 + y^2}$ *and* $\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}}$ and transform the above orbital equation into the following Cartesian form $k_1 x^2 + y^2 + k_2 x + k_3 y + k_4 = 0$. Write down k_1, k_2, k_3 *and* k_4 in terms of α *and* ε . **(4 marks)**

(ii) The orbital equation in (c)(i) would represent a parabolic orbit if $k_1 = 0$. Show that this condition is equivalent to the condition that $E = 0$. **(3 marks)**

(iii) The orbital equation in (c)(i) would represent a circular orbit if $k_1 = 1$. Show that this condition is equivalent to the condition that $E = -\frac{\mu k^2}{2 l^2} \equiv V_{min}$.

(3 marks)

(b) If an earth satellite of mass *500 kg* has a pure tangential speed $v_\theta \left(= r \dot{\theta}\right) = 9{,}000$ *m/s* at its near-earth-point *800 km* above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(4 marks)**

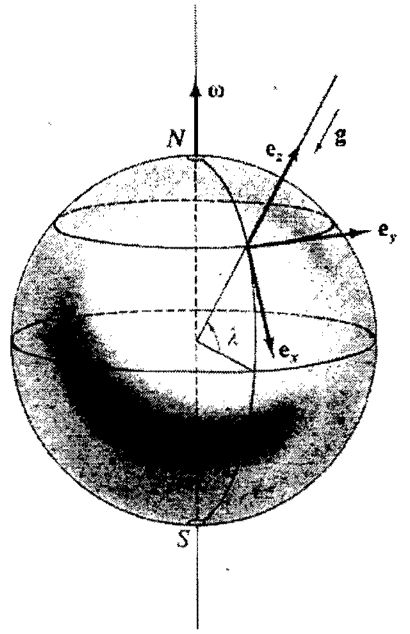
(ii) calculate the values of the eccentricity ε and show that the orbit is an elliptical orbit. Also calculate its semi-major axis a , semi-minor axis b and period. **(6 marks)**

(iii) what should be the value of the v_θ at the same given near-earth-point such that the satellite would maintain a circular orbit ? **(3 marks)**

(Hint : $E = \frac{1}{2} \mu v_\theta^2 - \frac{k}{r} \xrightarrow{\text{circular orbit}} -\frac{k}{2r}$)

(iv) Determine v_θ at the same given near-earth-point such that the satellite maintains a parabolic orbit ? **(2 marks)**

Question four



(a) If a person, near the earth surface at a northern latitude λ, fired a bullet of speed v_0 at a target situated at his north direction ($-\vec{e}_x$ *direction*) of distance L away from him. Assuming he has a perfect rifle and the time T for the bullet hitting the target is short and $T \approx \dfrac{L}{v_0}$ (i.e., neglecting the gravitational bending and assuming the bullet is moving along - x direction with constant speed v_0). The bullet will miss the target by a distance d resulting from the Coriolis force $\left(-2\, m\, \vec{\omega} \times \vec{v}_r\right)$.

(i) Show that

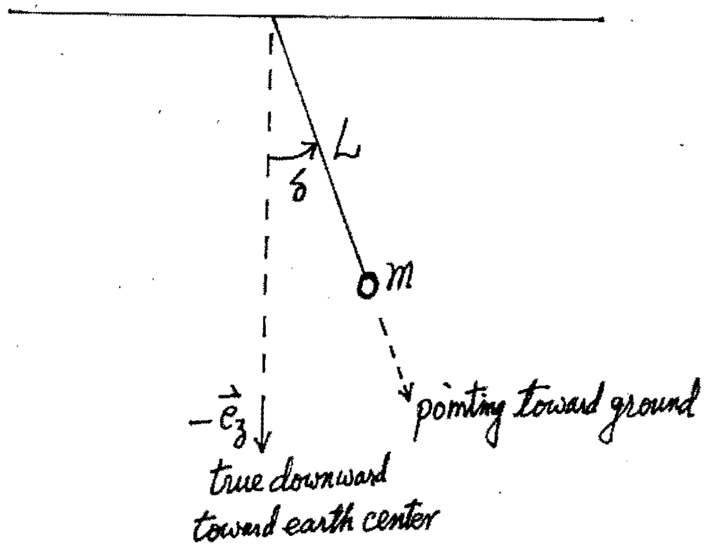
$$d = \frac{\omega L^2}{2 v_0} \sin(\lambda)$$ **(8 marks)**

(Hint: $\vec{a}_{eff} \approx -2\, \vec{\omega} \times \vec{v}_r$, $\vec{v}_r \approx \vec{e}_x\,(-v_0)$, $\vec{\omega} = \vec{e}_x\,(-\omega \cos(\lambda)) + \vec{e}_z\,(\omega \sin(\lambda))$)

(ii) The earth makes one cycle of self-rotation in a day, calculate the value of the earth's self-rotational speed ω in terms of *rad/s*. Then calculate the value of d if $v_0 =$ *2000 m/s* , $L =$ *3000 m* & $\lambda = 30^0$ **(3 marks)**

(b) Referring to the diagram above and considering the body coordinate system (x, y, z) has the same origin as the earth's fixed inertial system, i.e., center of the earth. Hanging a motionless simple pendulum of length L and mass m near the earth surface at a northern latitude λ, the pendulum is supposed to pointing direct downward along $-\vec{e}_z$ direction. Show that the pendulum is pointing toward a direction not exactly along $-\vec{e}_z$ direction, i.e., true downward direction pointing toward the earth center, but pointing toward the ground with a small angular deviation of δ made with the true downward direction resulting from the centrifugal force $(-m\, \vec{\omega} \times (\vec{\omega} \times \vec{r}))$ as shown in the following diagram.

Question four (continued)



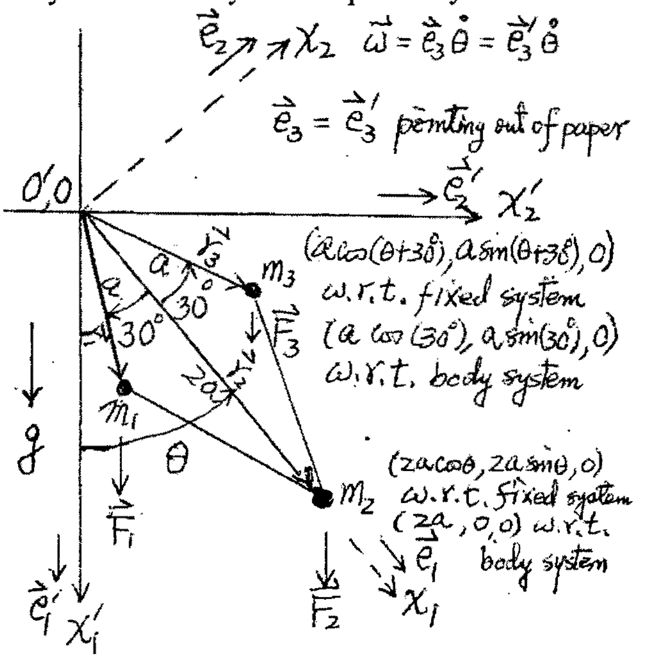
(i) Show that $\delta \approx \dfrac{\omega^2 r_E \cos(\lambda)\sin(\lambda)}{g - \omega^2 r_E \cos^2(\lambda)}$ **(10 marks)**

(Hint : $\vec{F}_{eff} \approx \vec{e}_z\,(-m\,g) - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$ & $\vec{r} \approx \vec{e}_z\,(r_E)$)

(ii) The maximum value of δ happens at $\lambda = 45^0$. Calculate this maximum value (in unit of radian) and then convert it to degrees. **(4 marks)**

Question five

Three particles of the same mass $m \; (= m_1 = m_2 = m_3)$ joined by several weightless rods to form a rigid body and is allowed to rotate about a fixed pivot point (chosen to be the origin $0'$ or 0). Let $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$ *and* $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the unit vectors of the fixed (i.e., inertial system) and body coordinate system respectively as shown in the figure below.



Asume the above rigid body only rotates on the $x_1{}' - x_2{}'$ plane with the angular velocity $\vec{\omega} = \dot{\vec{\theta}} = \vec{e}_3{}'\,\dot{\theta} = \vec{e}_3\,\dot{\theta}$ and the only force acting on the system is gravitational with $\vec{g} = \vec{e}_1{}'\, g$.

(a) (i) Find the three components of the total torque $\vec{N} \left(= \sum_{i=1}^{3} \vec{r}_i \times \vec{F}_i \right)$ on the rigid body along $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$ directions . In other words, write $\vec{N} = \vec{e}_1{}'\,N_1{}' + \vec{e}_2{}'\,N_2{}' + \vec{e}_3{}'\,N_3{}'$ and find $N_1{}'$, $N_2{}'$ & $N_3{}'$. Show that $N_1{}' = 0 = N_2{}'$ & $N_3{}' = -2\,m\,g\,a\sin(\theta)\left(1 + \cos(30^0)\right)$ **(4 marks)**

(Hint : $\begin{cases} \vec{r}_1 = \vec{e}_1{}'\,a\cos(\theta - 30^0) + \vec{e}_2{}'\,a\sin(\theta - 30^0) \\ \vec{r}_2 = \vec{e}_1{}'\,2\,a\cos(\theta) + \vec{e}_2{}'\,2\,a\sin(\theta) \\ \vec{r}_3 = \vec{e}_1{}'\,a\cos(\theta + 30^0) + \vec{e}_2{}'\,a\sin(\theta + 30^0) \end{cases}$ & $\begin{cases} \vec{F}_1 = \vec{e}_1{}'\,m\,g \\ \vec{F}_2 = \vec{e}_1{}'\,m\,g \\ \vec{F}_3 = \vec{e}_1{}'\,m\,g \end{cases}$ and

$\sin(A - B) + \sin(A + B) = 2\sin(A)\cos(B)$)

(ii) Find the three components of the total torque $\vec{N} \left(= \sum_{i=1}^{3} \vec{r}_i \times \vec{F}_i \right)$ on the rigid body along $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ directions . In other words, write $\vec{N} = \vec{e}_1\,N_1 + \vec{e}_2\,N_2 + \vec{e}_3\,N_3$ and find N_1 , N_2 & N_3 . Show that $N_1 = 0 = N_2$ & $N_3 = -2\,m\,g\,a\sin(\theta)\left(1 + \cos(30^0)\right)$ **(4 marks)**

Question five (continued)

(Hint: $\begin{cases} \vec{r}_1 = \vec{e}_1\, a\cos(30^0) - \vec{e}_2\, a\sin(30^0) \\ \vec{r}_2 = \vec{e}_1\, 2a \\ \vec{r}_3 = \vec{e}_1\, a\cos(30^0) + \vec{e}_2\, a\sin(30^0) \end{cases}$ & $\begin{cases} \vec{F}_1 = \vec{e}_1\, m\,g\cos(\theta) - \vec{e}_2\, m\,g\sin(\theta) \\ \vec{F}_2 = \vec{e}_1\, m\,g\cos(\theta) - \vec{e}_2\, m\,g\sin(\theta) \\ \vec{F}_3 = \vec{e}_1\, m\,g\cos(\theta) - \vec{e}_2\, m\,g\sin(\theta) \end{cases}$)

(b) Referring to the fixed (inertial) coordinate system, the equation for pure rotational motion is $\dot{\vec{L}} = \vec{N} \quad \cdots \quad (1)$ where $\vec{L} = \sum_{i=1}^{3} \vec{r}_i \times \vec{p}_i$ & $\vec{p}_i = m\,\dot{\vec{r}}_i \ (i = 1,2,3)$.

(i) Find the three components of the total angular momemtum $\vec{L}\left(= \sum_{i=1}^{3} m\,\vec{r}_i \times \dot{\vec{r}}_i \right)$ on the rigid body along $(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}')$ directions. In other words, write $\vec{L} = \vec{e}_1{}' L_1{}' + \vec{e}_2{}' L_2{}' + \vec{e}_3{}' L_3{}'$ and find $L_1{}'$, $L_2{}'$ & $L_3{}'$. Show that $L_1{}' = 0 = L_2{}'$ & $L_3{}' = 6\,m\,a^2\,\dot{\theta}$ **(5 marks)**

(ii) Use eq.(1) and the results from (a)(i) & (b)(i), deduce the following equation of rotational motion for the given rigid body and show that

$$\ddot{\theta} = -\frac{g}{3a}\left(1 + \cos(30^0)\right)\sin(\theta)$$ **(2 marks)**

(c) Referring to the body coordinate system, the equation for pure rotational motion is the following Euler equations that

$$\begin{cases} I_1\,\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\,\omega_3 = N_1 & \cdots \quad (2) \\ I_2\,\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\,\omega_1 = N_2 & \cdots \quad (3) \\ I_3\,\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\,\omega_2 = N_3 & \cdots \quad (4) \end{cases}$$

where $(\vec{e}_1 , \vec{e}_2 , \vec{e}_3)$ should be chosen properly such that the inertial tensor I is a diagonal matrix.

(i) Our choice of $(\vec{e}_1 , \vec{e}_2 , \vec{e}_3)$ here does make the inertial tensor a diagonal matrix. Find the three non-zero diagonal elements of I and show that

$$I_{1,1} = I_1 = \frac{1}{2}\,m\,a^2 \ , \ I_{2,2} = I_2 = \frac{11}{2}\,m\,a^2 \ \ \& \ \ I_{3,3} = I_3 = 6\,m\,a^2$$ **(6 marks)**

(ii) Since here one has $\omega_1 = 0 = \omega_2$ & $\omega_3 = \dot{\theta}$, use Euler equations and the results from (a)(ii) & (c)(i), deduce the equation of rotational motion for the given rigid body and show that it's the same as the one obtained in (b)(ii). **(4 marks)**

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad and\ reversely \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad and \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad and \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v]_{q,p} \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

radius of earth $r_E = 6.4 \times 10^6 \ m$

mass of earth $m_E = 6 \times 10^{24} \ kg$

earth attractive potential $\equiv -\frac{k}{r}$ *where* $k = G\, m\, m_E$

$$\varepsilon = \sqrt{1 + \frac{2 E l^2}{\mu k^2}} \quad \{(\varepsilon = 0, circle), (0 < \varepsilon < 1, ellipse), (\varepsilon = 1, parabola), \cdots\}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \approx m_1 \quad if \quad m_2 >> m_1$$

For elliptical orbit, i.e., $0 < \varepsilon < 1$, *then*

$$\begin{cases} semi-major \ \ a = \dfrac{k}{2|E|} \\ semi-minor \ \ b = \dfrac{l}{\sqrt{2\mu|E|}} \\ period \ \ \tau = \dfrac{2\mu}{l}(\pi a b) \\ r_{min} = a(1-\varepsilon) \ \& \ r_{max} = a(1+\varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r \dot{r} + \vec{e}_\theta r \dot{\theta} \\ \vec{a} = \vec{e}_r (\ddot{r} - r\dot{\theta}^2) + \vec{e}_\theta (2\dot{r}\dot{\theta} + r\ddot{\theta}) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,1} \, x_{\alpha,2} & -\sum_\alpha m_\alpha \, x_{\alpha,1} \, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,2} \, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,2} \, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,3} \, x_{\alpha,1} & -\sum_\alpha m_\alpha \, x_{\alpha,3} \, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$