UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2015/2016

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P320 CLASSICAL MECHANICS

Question one

(a) For a certain dynamical system the kinetic energy T and potential energy V are given by

$$T = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_1\,\dot{q}_2 + \dot{q}_2^2\right)$$

$$V = \frac{3}{2}q_2^2$$

where q_1 , q_2 are the generalized coordinates.

(i) Write down Lagrange's equations of motion and show that

$$\begin{cases} \ddot{q}_1 + \frac{1}{2}\ddot{q}_2 = 0 \\ \frac{1}{2}\ddot{q}_1 + \ddot{q}_2 = -3\,q_2 \end{cases}$$

(4 marks)

(ii) Eliminate $\ddot{q}_1$ from the Lagrange's equations of motion in (a)(i) to obtain a differential equation for q_2 and then show that the general solution of q_2 can be written as $q_2 = k_1\cos(2t) + k_2\sin(2t)$ where k_1 *and* k_2 are two arbitrary constants . **(4 marks)**

(iii) Substitute the general solution of q_2 in (a)(ii) back to any one equation in (a)(i) and integrate twice to show that the general solution of q_1 can be written as

$q_1 = -\frac{k_1}{2}\cos(2t) - \frac{k_2}{2}\sin(2t) + k_3\,t + k_4$ where k_3 *and* k_4 are another two arbitrary constants . **(4 marks)**

(iv) Write down their canonical momenta p_{q_1} & p_{q_2} and then deduce that

$$\begin{cases} \dot{q}_1 = \frac{4}{3}p_{q_1} - \frac{2}{3}p_{q_2} \\ \dot{q}_2 = -\frac{2}{3}p_{q_1} + \frac{4}{3}p_{q_2} \end{cases}$$

(5 marks)

(b) (i) Use $H = T + V$ and the results in (a)(iv) to write down the Hamiltonian of the system and deduce that

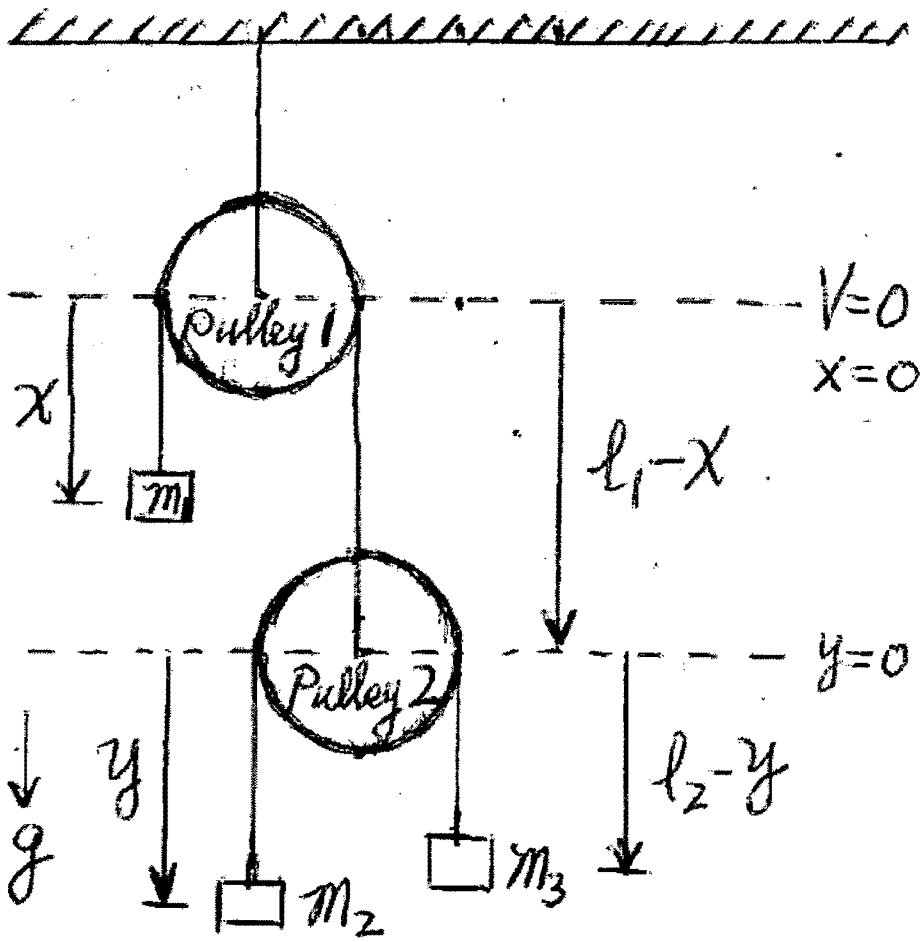
$$H = \frac{2}{3}\left(p_{q_1}{}^2 + p_{q_1}{}^2 - p_{q_1}\,p_{q_2}\right) + \frac{3}{2}q_2^2$$

(4 marks)

(ii) From the Hamiltonian in (b)(i), write down the equations of motion of the system. **(4 marks)**

Question two

(a) Consider the double pulley system as shown in the following diagram :



Assume the pulleys are massless and let l_1 & l_2 be the length of rope hanging freely from *pulley 1* and *pulley 2* respectively. Assume the pulley system subject only to gravitational force with zero gravitational potential set at $x = 0$.

(i) Write down the Lagrangian L for the given system and show that it can be simplified to the following expression :

$$L = \frac{1}{2}(m_1 + m_2 + m_3)\dot{x}^2 + \frac{1}{2}(m_2 + m_3)\dot{y}^2 + (m_3 - m_2)\dot{x}\,\dot{y} + m_1\,g\,x + m_2\,g\,(l_1 - x + y) + m_3\,g\,(l_1 - x + l_2 - y)$$

(8 marks)

(ii) Write down Lagrange's equations of motion and show that

$$\begin{cases}(m_1 + m_2 + m_3)\ddot{x} + (m_3 - m_2)\ddot{y} = (m_1 - m_2 - m_3)\,g \\ (m_3 - m_2)\ddot{x} + (m_3 + m_2)\ddot{y} = (m_2 - m_3)\,g\end{cases}$$

(4 marks)

(b) If H denotes the Hamiltonian function and L is the Lagrangian function, use the definition $H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$ (where p_α *and* q_α $(\alpha = 1, 2, \cdots, n)$ are the generalized momenta and coordinates respectively, i.e., $H = H(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$, $L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$, $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$ *and* $\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$) to show that

(i) $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$ $\quad \alpha = 1, 2, \cdots, n$ **(4 marks)**

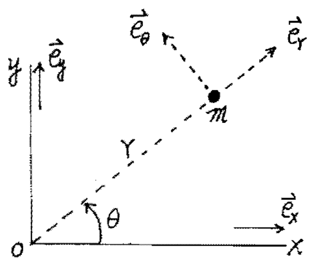
(ii) $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$ $\quad \alpha = 1, 2, \cdots, n$ **(4 marks)**

(iii) $\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}$ where $F = F(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$ and

$[F, H] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right)$. **(5 marks)**

Question three

Consider a particle of mass m acted on by an attractive central force of $\vec{F} = -\ \vec{e}_r \dfrac{k}{r^n}$, where k is a positive constant and $n > 1$, and moving in a 2-D plane described by the plane polar coordinates as shown in the diagram below.



The kinetic energy of this particle in this plane polar coordinate is
$T = \dfrac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right)$ ...... (1) .

(i) From $V = -\int_{r_0}^{r} \vec{F} \bullet d\vec{l}$ *where* $d\vec{l} = d\vec{r} = \vec{e}_r\, d r + \vec{e}_\theta\, r\, d\theta$ & $r_0 \to \infty$, find the potential energy V of this particle in this plane polar coordinate under the given force $\vec{F} = -\ \vec{e}_r \dfrac{k}{r^n}$ where k is a constant. Show that

$$V = -\left(\frac{k}{(n-1)\,r^{n-1}}\right) \quad \cdots\cdots \quad (2)$$ **(3 marks)**

(ii) Write down the Lagrange equations of motion for this system and show that

$$\begin{cases} m\,\ddot{r} = \left(m\,r\,\dot{\theta}^2 - \dfrac{k}{r^n}\right) & \cdots\cdots \quad (3) \\ \dfrac{d}{dt}\left(m\,r^2\,\dot{\theta}\right) = 0 & \cdots\cdots \quad (4) \end{cases}$$ **(5 marks)**

(iii) Write its $(r\,,\theta)$ respective momentums, i.e., p_r & p_θ , and show that the angular momentum p_θ is a constant and then eq.(1) & eq.(3) can be rewritten in terms of this constant p_θ as $T = \dfrac{m\,\dot{r}^2}{2} + \dfrac{p_\theta^{\ 2}}{2\,m\,r^2}$ ...... (1)' and $m\,\ddot{r} = \left(\dfrac{p_\theta^{\ 2}}{m\,r^3} - \dfrac{k}{r^n}\right)$ ...... (3)' respectively. **(4 mark)**

(iv) Multiply eq.(1)' by $d\,r$ and show that it can be rewritten as $d(T + V) = 0$ and thus this implies the total energy $(T + V)$ is also a constant. **(5 marks)**

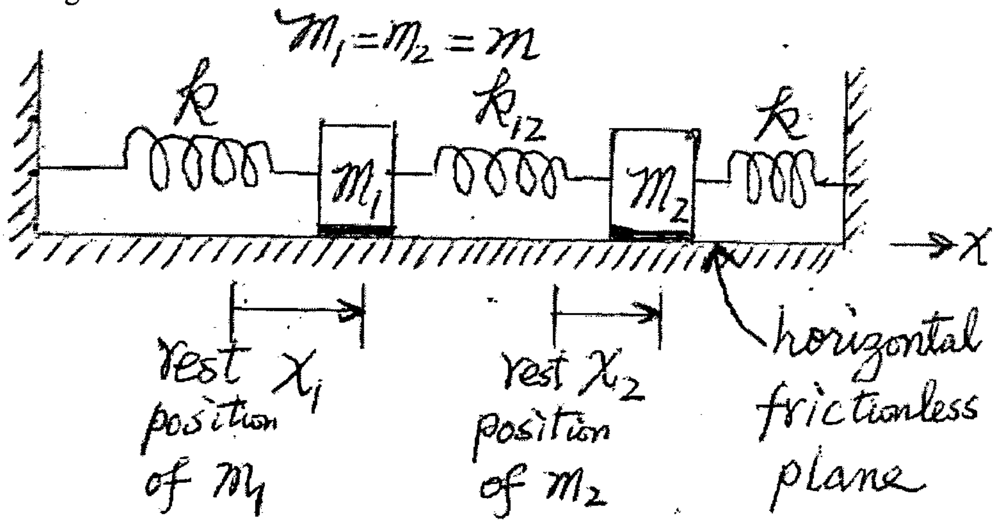
(Hint : $\ddot{r}\,d\,r = \dfrac{d\,\dot{r}}{d\,t}\,d\,r = d\,\dot{r}\,\dfrac{d\,r}{d\,t} = \dot{r}\,d\,\dot{r} = d\left(\dfrac{\dot{r}^2}{2}\right)$)

(v) For circular orbits , i.e., r is a constant , find a relation between the kinetic and potential energies and show that $T = -\dfrac{n-1}{2}V$. **(8 marks)**

(Hint : Use eq.(1)' , eq.(3)' , eq.(2) and $\dot{r} = 0$ & $\ddot{r} = 0$)

Question four

(a) Two identical simple harmonic oscillators of mass m and spring constant k are joined by a spring of spring constant k_{12} and allowed to oscillate on a horizontal frictionless plane along $x-direction$ as shown below :



where x_1 & x_2 are the displaced lengths from the rest positions of m_1 & m_2 respectively. The Lagrangian for the system can be written as:

$$L = \frac{1}{2} m \dot{x}_1^{\,2} + \frac{1}{2} m \dot{x}_2^{\,2} - \frac{1}{2} k x_1^{\,2} - \frac{1}{2} k_{12} (x_1 - x_2)^2 - \frac{1}{2} k x_2^{\,2}$$

(i) Write down the equations of motion and deduce that

$$\begin{cases} \ddot{x}_1 = -\left(\dfrac{k+k_{12}}{m}\right) x_1 + \left(\dfrac{k_{12}}{m}\right) x_2 \\ \ddot{x}_2 = \left(\dfrac{k_{12}}{m}\right) x_1 - \left(\dfrac{k+k_{12}}{m}\right) x_2 \end{cases}$$

(4 marks)

(ii) Set $x_1 = \hat{X}_1 e^{i\omega t}$ *and* $x_2 = \hat{X}_2 e^{i\omega t}$ (where $\hat{X}_1$ *and* $\hat{X}_2$ are constants) and deduce from the equations in (a)(i) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(\dfrac{k+k_{12}}{m}\right) & \left(\dfrac{k_{12}}{m}\right) \\ \left(\dfrac{k_{12}}{m}\right) & -\left(\dfrac{k+k_{12}}{m}\right) \end{pmatrix}$$

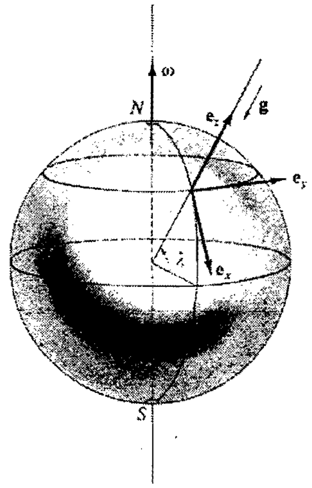
(4 marks)

(iii) Show that the eigenfrequencies ω of this coupled system are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \& \quad \omega_2 = \sqrt{\frac{k+2k_{12}}{m}}$$

(4 marks)

(b)



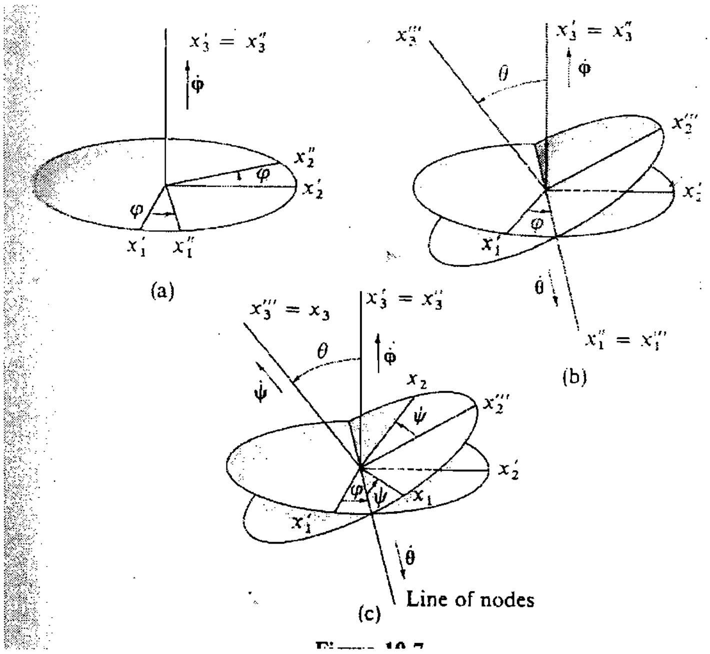
If a particle is projected vertically upward with an initial speed v_0 to a height h above a point on the earth's surface at northern latitude λ , show that it strikes the ground at a point $\frac{4}{3}\,\omega\cos(\lambda)\sqrt{\frac{8\,h^3}{g}}$ to the west of the initial throwing point. Neglect air resistance and only consider small vertical height. **(13 marks)**

(Hint :
$\vec{a}_{eff} \approx \vec{e}_z\,(-g) - 2\,\vec{\omega}\times\vec{v}_r$, $\vec{v}_r \approx \vec{e}_z\,(v_0 - g\,t)$, $\vec{\omega} = \vec{e}_x\,(-\omega\cos(\lambda)) + \vec{e}_z\,(\omega\sin(\lambda))$
and $v_0 = \sqrt{2\,g\,h}$, $(\textit{total time for the given motion}) = \frac{2\,v_0}{g}$)

Question five

(a) The fixed (or inertia) coordinate system X' shares the same origin with the body coordinate system X such that only rotational motion is considered. The rotational velocity $\vec{\omega}$ of the body system with respect to the fixed system are breaking down into three independent angular velocities , i.e., $\vec{\omega} = \vec{\dot{\varphi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}$ where (φ, θ, ψ) are Euler's angles . We use two intermediate coordinate systems X'' & X''' to bridge between X' & X systems such that $X'' = \lambda_\varphi X'$, $X''' = \lambda_\theta X''$ & $X = \lambda_\psi X'''$ as shown in the following diagram , where

$$\lambda_\varphi = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}, \lambda_\psi = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



(i) Since the direction of $\vec{\dot{\varphi}}$ is along x_3'*–axis* (which is the same as x_3''*–axis*) with the magnitude of $\dot{\varphi}$ thus $(\vec{\dot{\varphi}})'' = \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}$ *in* X'' *system* , show that $\vec{\dot{\varphi}}$ in X system(i.e., the body system) is $(\vec{\dot{\varphi}}) = \begin{pmatrix} \dot{\varphi}\sin(\theta)\sin(\psi) \\ \dot{\varphi}\sin(\theta)\cos(\psi) \\ \dot{\varphi}\cos(\theta) \end{pmatrix}$ *in X system*. In other words, show that

$$\begin{pmatrix} \dot{\varphi}\sin(\theta)\sin(\psi) \\ \dot{\varphi}\sin(\theta)\cos(\psi) \\ \dot{\varphi}\cos(\theta) \end{pmatrix} = \lambda_\psi \lambda_\theta \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix}$$

(6 marks)

Question five (continued)

(ii) Since the direction of $\vec{\dot{\theta}}$ is along x_1'' *–axis* (which is the same as x_1''' *–axis*) with the magnitude of $\dot{\theta}$ thus $\left(\vec{\dot{\theta}}\right)''' = \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$ *in* X''' *system*, show that $\vec{\dot{\theta}}$ in X system(i.e., the body system) is $\left(\vec{\dot{\theta}}\right) = \begin{pmatrix} \dot{\theta}\cos(\psi) \\ -\dot{\theta}\sin(\psi) \\ 0 \end{pmatrix}$ *in X system*. In other words, show that $\begin{pmatrix} \dot{\theta}\cos(\psi) \\ -\dot{\theta}\sin(\psi) \\ 0 \end{pmatrix} = \lambda_\psi \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$. **(4 marks)**

(Note : Since the direction of $\vec{\dot{\psi}}$ is along x_3''' *–axis* (which is the same as x_3 *–axis*) with the magnitude of $\dot{\psi}$ thus $\left(\vec{\dot{\psi}}\right) = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$ *in X system*. Then the rotational velocity $\vec{\omega} = \vec{\dot{\varphi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}$ in X system (i.e., body system) can be written in terms of Euler's angles as $(\vec{\omega}) \equiv \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\varphi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \\ \dot{\varphi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi) \\ \dot{\varphi}\cos(\theta) + \dot{\psi} \end{pmatrix}$)

(b) (i) By proper choice of the orientation of the body coordinate system, the inertia tensor I (i.e., rotational mass) of a rigid body can be in the form of a diagonal matrix, i.e., $I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$, thus its rotational kinetic energy is

$T_{rot} = \frac{1}{2} I_1\, \omega_1^{\,2} + \frac{1}{2} I_2\, \omega_2^{\,2} + \frac{1}{2} I_3\, \omega_3^{\,2}$.

Consider a pure rotational motion of the rigid body under no external force, then its Lagrangian is $L = T_{rot} = \frac{1}{2} I_1\, \omega_1^{\,2} + \frac{1}{2} I_2\, \omega_2^{\,2} + \frac{1}{2} I_3\, \omega_3^{\,2}$ where ω_1 , ω_2 & ω_3 are functions of $(\varphi, \theta, \psi, \dot{\varphi}, \dot{\theta}, \dot{\psi})$ as those given in (a)(ii).

Write down the Lagrange equation of motion for ψ, i.e., $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{\partial L}{\partial \psi}$, and show that it can be simplified to

$(I_1 - I_2)\,\omega_1\,\omega_2 - I_3\,\dot{\omega}_3 = 0$. **(13 marks)**

Question five (continued again)

(ii) From $\left(I_1 - I_2\right)\omega_1\,\omega_2 - I_3\,\dot{\omega}_3 = 0$ derived in (b)(i), the other two independent equations for the above rigid body's motion can be simply written down directly as $\left(I_2 - I_3\right)\omega_2\,\omega_3 - I_1\,\dot{\omega}_1 = 0$ & $\left(I_3 - I_1\right)\omega_3\,\omega_1 - I_2\,\dot{\omega}_2 = 0$ without going through similar derivations as done in (b)(i). Explain briefly why so ?

(2 marks)

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad \text{and reversely} \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \text{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\,m^2}{kg^2}$$

$$\text{radius of earth} \quad r_E = 6.4 \times 10^6 \ m$$

$$\text{mass of earth} \quad m_E = 6 \times 10^{24} \ kg$$

$$\text{earth attractive potential} \equiv -\frac{k}{r} \quad \text{where} \quad k = G\, m\, m_E$$

$$\varepsilon = \sqrt{1 + \frac{2\,E\,l^2}{\mu\, k^2}} \quad \{(\varepsilon = 0, \text{circle}), (0 < \varepsilon < 1, \text{ellipse}), (\varepsilon = 1, \text{parabola}), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_2 >> m_1$$

$$\text{For elliptical orbit, i.e., } 0 < \varepsilon < 1, \text{ then} \begin{cases} \text{semi-major} \quad a = \dfrac{k}{2\,|E|} \\ \text{semi-minor} \quad b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ \text{period} \quad \tau = \dfrac{2\,\mu}{l} (\pi\, a\, b) \\ r_{\min} = a\,(1 - \varepsilon) \ \& \ r_{\max} = a\,(1 + \varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r \left(\ddot{r} - r\,\dot{\theta}^2\right) + \vec{e}_\theta \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,1} \, x_{\alpha,2} & -\sum_\alpha m_\alpha \, x_{\alpha,1} \, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,2} \, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,2} \, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,3} \, x_{\alpha,1} & -\sum_\alpha m_\alpha \, x_{\alpha,3} \, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_r$$