UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION: 2015/2016

TITLE OF THE PAPER: QUANTUM MECHANICS

COURSE NUMBER: P342

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

- ANSWER ANY FOUR OUT THE FIVE QUESTIONS.
- EACH QUESTION CARRIES 25 MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN PAGE 2 WHEN NECESSARY.

THIS PAPER HAS 7 PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THIS PAGE UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR

Useful Formulas

Time-dependent Schrodinger equation: $i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t)$

Time-independent Schrodinger equation: $\hat{H}\psi(x) = E\psi(x)$

Hamiltonian operator : $\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + V(x)$

Momentum operator $\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x)$

Probability current $J(x,t) = \frac{i\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right)$

Fourier transform $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}dk$

Inverse Fourier transform $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx$

$\int_{-\infty}^{\infty} x^{2n}e^{-ax^2}dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}$, n =0, 1, 2, 3, ...

$\int_{-\infty}^{\infty} x^{2n+1}e^{-ax^2}dx = 0$, n =0, 1, 2, 3, ...

Heisenberg uncertainty $\Delta x\Delta p \geq \hbar/2$

Uncertainty of a quantity $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$

Infinite potential well:

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{elsewhere} \end{cases}$$

$E_n = \frac{\hbar^2\pi^2n^2}{2ma^2}$ and $u_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$

Trigonometry:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Question 1

Consider an electron in a one-dimensional infinite potential well of the width a:

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{elsewhere} \end{cases}$$

Suppose the initial normalized wave function of the electron is given as

$$\psi(x, t = 0) = A \cos^4\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right).$$

(a) What is the wave function at later time t, $\psi(x, t)$?

[8 marks]

(b) Calculate the normalization constant A.

[4 marks]

(c) What is the probability that an energy measurement yields E_3 where

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \ ?$$

[3 marks]

(d) If this measurement is repeated on many identical systems, what is the average value of the energy that will be found?

[3 marks]

(e) Using the uncertainty principle, estimate the order of the electron's speed in this well as a function of the speed of light (the rest mass of the electron is $m_e c^2 \simeq 0.511$ MeV, $\hbar c \simeq 197$ MeV$\cdot$ fm, $a = 8 \times 10^4$fm).

[7 marks]

Question 2

At time $t = 0$ a particle of mass m trapped in an infinite square well of width L is in a superposition of ground state and the fourth excited state,

$$\psi_s(x,0) = N[3u_1(x) - 2iu_5(x)]$$

where the $u_n(x)$ are correctly-normalized energy eigenstates with energies E_n.

(a) Which of the following values of N give a properly normalized wavefunction?

i) $\frac{1}{\sqrt{5}}$ ii) $\frac{i}{5}$ iii) $\frac{-i}{\sqrt{13}}$ iv) $\frac{1}{13}$ v) None of these

[5 marks]

(b) Given the wavefunction ψ_s, what is the probability that the energy is E_5 at $t = 0$? Explain.

i) 0 ii) 3/5 iii) 4/13 iv) 9/25 v) 6/13

[5 marks]

(c) Let $\phi_{nlm}(\mathbf{r})$ denote the ortho-normalized energy eigenfunctions of the Coulomb potential with principal quantum number n and angular momentum quantum numbers l and m. Consider the state $\psi(\mathbf{r}) = C[\phi_{100}(\mathbf{r}) + 4i\phi_{210}(\mathbf{r}) - 2\sqrt{2}\phi_{21-1}(\mathbf{r})]$. The expectation value of the z-component of the particle angular momentum $\hat{L}_z$ is

i) 0 ii) $80\hbar^2/25$ iii) 1/5 iv) $-8\hbar/25$ v) None of these

[5 marks]

(d) Let ϕ_n be the properly-normalized n^{th} energy eigenfunction of the harmonic oscillator, and let $\psi = \hat{a}\hat{a}^\dagger\phi_n$. Which of the following is equal to ψ?

i) ϕ_n ii) $n\phi_{n-1}$ iii) $(n+1)\phi_n$ iv) $n\phi_{n+1}$ v) None of these

[5 marks]

(e) A particle of mass m and charge q is accelerated across a potential difference V to a non-relativistic velocity. What is the de Broglie wavelength λ of this particle?

i) $\frac{m}{2hqV}$ ii) $\frac{qV}{\sqrt{2mqV}}$ iii) $\frac{h}{\sqrt{2mqV}}$ iv) $\frac{m}{\hbar\sqrt{2qV}}$ v) Something Else

[5 marks]

NB: For full marks please support your answer.

Question 3

The evolution of a quantum particle is in a state $|\psi, t\rangle$ which is a solution to the time dependent Schrodinger equation

$$i\hbar\frac{d}{dt}|\psi, t\rangle = \hat{H}|\psi, t\rangle$$

then the expectation value of any operator $\hat{O}$ satisfies:

$$\frac{d}{dt}\langle\psi, t|\hat{O}|\psi, t\rangle = \frac{i}{\hbar}\langle\psi, t|[\hat{H}, \hat{O}]|\psi, t\rangle$$

(a) A classical particle moving in a potential $V(x)$ obeys the following relations:

$$\frac{dx}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -\frac{dV(x)}{dx}$$

Show that for a quantum particle with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

satisfies the analogous relations:

$$\frac{d}{dt}\langle\psi, t|\hat{x}|\psi, t\rangle = \frac{1}{m}\langle\psi, t|\hat{p}|\psi, t\rangle$$

$$\frac{d}{dt}\langle\psi, t|\hat{p}|\psi, t\rangle = -\langle\psi, t|\frac{dV(\hat{x})}{d\hat{x}}|\psi, t\rangle$$

[10 marks]

(b) Consider a particle of mass m which moves under the influence of gravity; the particle's Hamiltonian is $\hat{H} = \hat{P}_y^2/2m - mg\hat{y}$, where g is the acceleration due to gravity, $g = 9.8$ m/s^2.

(i) Calculate

$$\frac{d\langle\hat{y}\rangle}{dt}, \quad \frac{d\langle\hat{P}_y\rangle}{dt}, \quad \text{and} \frac{d\langle\hat{H}\rangle}{dt}.$$

[9 marks]

(ii) Solve the equation $d\langle\hat{y}\rangle/dt$ and the obtain $\langle\hat{y}\rangle(t)$, such that $\langle\hat{y}\rangle(0) = h$ and $\langle\hat{P}_y\rangle(0) = 0$. Compare the result with the classical relation $y(t) = -\frac{1}{2}gt^2+h$.

[6 marks]

Question 4

(a) An electron in the Coulomb field of a proton is in a state described by the wavefunction

$$\psi(\vec{r}) = \frac{1}{\sqrt{10}}(2\phi_{100} + \phi_{210} + \sqrt{2}\phi_{211} + \sqrt{3}\phi_{21-1})$$

where the subscripts are the values of the quantum number n, l, and m, respectively. Note that $L_z\phi_{nlm} = m\hbar\phi_{nlm}$.

(i) If the z-component of the electron's angular momentum L_z were measured, what values would one obtain, and with what probabilities?

[6 marks]

(ii) Determine the expectation value of the z-component of the electron's angular momentum L_z.

[5 marks]

(iii) If a measurement of the electron's energy were carried out what values would be found, and with what probabilities? Note that $\hat{H}\psi_{nlm} = E_n\hbar\psi_{nlm}$, where $E_n = -13.6eV/n^2$.

[4 marks]

(iv) Determine the mean value of the electron's energy $\langle E \rangle$.

[4 marks]

(b) Consider a hydrogen atom which is in its ground state; the ground state wave function is given by

$$\psi(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where a_0 is the Bohr radius. Find the most probable distance between the electron and proton when the hydrogen atom is its ground state.

[6 marks]

Question 5

A particle of mass m in a one dimensional simple harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$ has the initial wavefunction

$$|\psi, t=0\rangle = N(|1\rangle - \sqrt{5}i|2\rangle + \sqrt{3}i|4\rangle)$$

where $|n\rangle$, $n = 0, 1, ...,$ are the harmonic oscillator energy eigenstate, satisfying $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ where $\hat{a}^\dagger$ and $\hat{a}$ are the raising and lowering operators respectively. In term of $\hat{a}^\dagger$ and $\hat{a}$, the Hamiltonian operator of this system $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

(a) Find N such that $|\psi, t=0\rangle$ is properly normalized.

[3 marks]

(b) What is the wavefunction at a later time, $|\psi, t\rangle$?

[3 marks]

(c) Compute the expectation value of the energy $\langle\psi, t|\hat{H}|\psi, t\rangle$?

[5 marks]

(d) If a measurement of the electron's energy were carried out what values could be found and with what probabilities?

[5 marks]

(e) What is the most probable result for a measurement of the particle's energy?

[2 marks]

(f) What is the expectation value for the particle's potential energy $\langle\psi, t|\frac{1}{2}m\omega^2\hat{x}^2|\psi, t\rangle$. *Hint:* $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$

[7 marks]