UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2015/2016

TITLE OF PAPER: STATISTICAL PHYSICS & THERMODYNAMICS

COURSE NUMBER: P461

TIME ALLOWED : THREE HOURS

ANSWER ANY **FOUR** QUESTIONS OUT OF THE FIVE QUESTIONS. ALL QUESTIONS CARRY EQUAL MARKS.

THIS PAPER CONTAINS EIGHT PAGES INCLUDING TWO APPENDICES. IT SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION ONE

(a) (i) What is meant by *statistical weight* of a system of particles? (2 marks)

(ii) State its significance as regards the properties of the system. (2 marks)

(b) (i) A system has 6 distinguishable particles arranged in 2 non-degenerate energy levels. What are the possible macrostates? (2 marks)

(ii) Find the number of microstates corresponding to each macrostate and hence determine the most probable configuration.

Given: $W = N!\prod\left(\frac{g_s^{n_s}}{n_s!}\right)$ (8 marks)

(c) (i) Define *density of states* in phase space. (2 marks)

(ii) Derive an expression for the volume element in phase space in terms of energy and show that the density of states can be expressed as

$$g(\varepsilon)d\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}d\varepsilon$$

where the symbols have their usual meanings. (9 marks)

QUESTION TWO

(a) Derive the Bose-Einstein distribution function $n_s = g_s e^{\alpha + \beta \varepsilon_s}$ for an isolated system of non-interacting particles in thermal equilibrium.

(11 marks)

(b) Assuming that the constant β in the above distribution function is equal to $-1/(kT)$, k being the Boltzmann constant, show that the entropy S of the system

$$S = k \ln W$$

where W is the maximum weight of the system.

(6 marks)

(c) Given that the partition function $Z = \sum_s g_s e^{\beta \varepsilon_s}$ show that the entropy of the system:

$$S = Nk \ln Z + \frac{E}{T}$$

where the symbols have their usual meanings. (8 marks)

QUESTION THREE

(a) Two systems of non-interacting and identical particles in thermal equilibrium have entropies S_1 and S_2, and weights W_1 and W_2. If the two systems are combined, what is

(i) the total entropy and (1 mark)

(ii) the total weight of the combined system? (1 mark)

Verify that your results agree with the equation $S = k \ln W$, where W is the maximum weight of the system.

(2 marks)

(b) Given that the density of states of a classical perfect gas is

$$g(\varepsilon)d\varepsilon = \frac{V}{h^3} 2\pi(2m)^{3/2} \varepsilon^{1/2} d\varepsilon$$

where the symbols have their usual meanings, show that the partition function of the gas

$$Z = \frac{V}{h^3}(2\pi mkT)^{3/2}$$ (7 marks)

(c) (i) Show that the pressure of the classical gas $P = NkT\left(\frac{\partial \ln Z}{\partial V}\right)_T$

(6 marks)

(ii) Hence derive the ideal gas equation: $PV = NkT$ (4 marks)

(iii) Show that the total energy of the above system $E = \frac{3}{2}kT$. (4 marks)

Given: $$E = NkT^2\left(\frac{\partial \ln Z}{\partial T}\right)_V$$

QUESTION FOUR

(a) (i) Obtain the mean energy of a quantum mechanical one-dimensional harmonic oscillator by setting up its partition function.

(8 marks)

(ii) "High temperature is the classical limit of quantum mechanics". Given that the mean energy of a classical oscillator is kT, verify the validity of this statement for the harmonic oscillator.

(4 marks)

(Given: $e^x = 1 + x/1! + x^2/2!$ for small values of x)

(b) (i) State the basic assumptions of Einstein's theory of specific heat of solids.

(3 marks)

(ii) Obtain an expression for the specific heat of solids based on these assumptions.

(6 marks)

(c) At low temperatures, the specific heat of a solid varies proportional to T^3. Does Einstein's theory agree with this? Explain briefly.

(4 marks)

QUESTION FIVE

(a) Define *Fermi energy.* (2 marks)

(b) The Fermi function of a system is given as

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_F)/kT} + 1}$$

(i) State the significance of this function as regards the distribution of the fermions in the system. (2 marks)

(ii) Compute the value of the Fermi function at 0 K for the cases

$\epsilon < \epsilon_F$ and $\epsilon > \epsilon_F$ (4 marks)

(iii) Sketch graphs of $f(\epsilon)$ versus ϵ for temperatures $T = 0$ K and $T > 0$ K, and state what physical meaning you can infer from them. (4 marks)

(c) Use Fermi-Dirac statistics to show that the contribution by free electrons in a metal toward its specific heat is proportional to the absolute temperature.

Given: Fermi energy $\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$

(8 marks)

(d) Calculate the Fermi energy of lithium having electron density of 5 x 10^{28} m^{-3}

(5 marks)

Appendix 1

Definite integrals

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty e^{-ax^2}\,x\,dx = \frac{1}{2a}$$

$$\int_0^\infty e^{-ax^2}\,x^2\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty e^{-ax^2}x^3\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty e^{-ax^2}\,x^4\,dx = \frac{3}{8a^2}\sqrt{\left(\frac{\pi}{a}\right)}$$

$$\int_0^\infty e^{-ax^2}\,x^5\,dx = \frac{1}{a^3}$$

$$\int_0^\infty \frac{x^3\,dx}{e^x - 1} = \frac{\pi^4}{15}$$

$$\int_0^\infty x^{1/2}\,e^{-\lambda x}\,dx = \frac{\pi^{1/2}}{2\lambda^{3/2}}$$

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1}\,dx = \frac{2.61\pi^{1/2}}{2}$$

$$\int_0^\infty \frac{x^4 e^x}{(e^x - 1)^2}\,dx = \frac{4\pi^4}{15}$$

Appendix 2

Physical Constants

Quantity	*symbol*	*value*
Speed of light	c	3.00×10^{8} m s^{-1}
Planck's constant	h	6.63×10^{-34} J s
Stefan - Boltzmann constant	k	1.38×10^{-23} J K^{-1}
Electronic charge	e	1.61×10^{-19} C
Mass of electron	m_e	9.11×10^{-31} kg
Mass of proton	m_p	1.67×10^{-27} kg
Gas constant	R	8.31 J mol^{-1} K^{-1}
Avogadro's number	N_A	6.02×10^{23}
Bohr magneton	μ_B	9.27×10^{-24} J T^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Stefan constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Atmospheric pressure		1.01×10^{5} N m^{-2}
Mass of ${}_2^4$He atom		6.65×10^{-27} kg
Mass of ${}_2^3$He atom		5.11×10^{-27} kg
Volume of an ideal gas at STP		22.4 L mol^{-1}