UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2016/2017

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P320 CLASSICAL MECHANICS

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Question one

(a) If H denotes the Hamiltonian function and L is the Lagrangian function, use the definition $H=\sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$ [where p_α *and* q_α $(\alpha = 1,2,\cdots,n)$ are the generalized momenta and coordinates respectively, i.e., $H = H(q_1,\cdots,q_n,p_1,\cdots,p_n,t)$, $L = L(q_1,\cdots,q_n,\dot{q}_1,\cdots,\dot{q}_n,t)$, $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$ *and* $\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$] to show that

(i) $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \alpha = 1,2,\cdots,n$ **(4 marks)**

(ii) $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \quad \alpha = 1,2,\cdots,n$ **(4 marks)**

(iii) $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ **(7 marks)**

(b) For a certain dynamical system the kinetic energy, T, and potential energy, V, are given by

$T = \dot{q}_1^2 + 2\,\dot{q}_1\,\dot{q}_2 + 3\,\dot{q}_2^2$

$V = 4\,q_1^2$

where q_1 , q_2 are the generalized coordinates.

(i) Find the momentum p_1 & p_2 of the system. **(2 marks)**

(ii) Use $H=\sum_{\alpha=1}^{2} p_\alpha \dot{q}_\alpha - L$ to find the Hamiltonian function of the system and show that $H = \frac{1}{8}\left(3\,p_1^{\,2} - 2\,p_1\,p_2 + p_2^{\,2}\right) + 4\,q_1^2$ **(8 marks)**

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Question two

The definition of the Poisson brackets are given as $[u,v]_{q,p} \equiv \sum_{\alpha=1}^{n}\left(\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right)$, or simply written as $[u,v]$, where q_α and p_α are the α^{th} generalized coordinate and momentum respectively.

(a) For any function $F(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n, t)$, prove that

$$\frac{dF}{dt} = [F,H] + \frac{\partial F}{\partial t}$$

where H is the Hamiltonian of the system, i.e., $H(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n, t)$

(5 marks)

(b) The three components of the angular momentum $\vec{l}\ (\equiv \vec{r} \times \vec{p})$ of a particle of mass m are given by $l_1 = q_2\, p_3 - q_3\, p_2$, $l_2 = q_3\, p_1 - q_1\, p_3$ *and* $l_3 = q_1\, p_2 - q_2\, p_1$ where $p_i = m\,\dot{q}_i$ $i = 1,2,3$. Show that

(i) $[l_1, l_2] = l_3$ **(5 marks)**

(ii) $[q_2, l_3] = q_1$ **(5 marks)**

(c) For an equation of the type $\frac{du}{dt} = [u,H]$ the specific solution of $u(t)$ is given by the following Taylor series expansion for the time t as

$$u(t) = u_0 + [u,H]_0\, t + [[u,H],H]_0 \frac{t^2}{2!} + [[[u,H],H],H]_0 \frac{t^3}{3!} + \cdots\cdots\cdots \qquad \cdots\cdots \quad (1)$$

where subscript 0 denotes the initial conditions at $t = 0$.

For a simple harmonic oscillator system described by $H = \frac{p^2}{2m} + \frac{1}{2}k\,x^2$, if the initial conditions are given as x_0 *and* p_0, use *eq.(1)* to deduce that

$$x(t) = x_0 + \frac{p_0}{m}t - \frac{k\,x_0}{2m}t^2 - \frac{k\,p_0}{6m^2}t^3 + \cdots\cdots$$

(10 marks)

Question three

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(a) Given the Lagrangian for the two-body central force system as :

$$L = T - V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}$$

where μ is the reduced mass of the system, k is a positive constant and (r, θ) are polar coordinates of the motion plane with its origin at the center of mass of the two-body system.

(i) Write down the Lagrange's equation for θ and show that the angular momentum l is conserved , i.e., deduce that

$\dot{\theta} = \frac{l}{\mu r^2}$ ...... (1) where l is a constant. **(3 marks)**

(ii) Write down the Lagrange's equation for r , with eq.(1) inserted, deduce that

$\mu \ddot{r} - \frac{l^2}{\mu r^3} + \frac{k}{r^2} = 0$ ...... (2) **(3 marks)**

(iii) Multiply eq.(2) by dr and use $\ddot{r}\, dr = \frac{d\dot{r}}{dt} dr = d\dot{r}\frac{dr}{dt} = \dot{r}\, d\dot{r} = d\left(\frac{\dot{r}^2}{2}\right)$ to show that the total energy $E\,(\equiv T + V)$ is conserved , i.e.,

$\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r} = const. \equiv E$ ...... (3) **(6 marks)**

(b) If an earth satellite of 500 kg mass is having a pure tangential speed $v_\theta = 8{,}000$ m/s at its near-earth-point 600 km above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(3 marks)**

(ii) calculate the values of the eccentricity, ε , and show that the orbit is an elliptical orbit. Also calculate its period. **(6 marks)**

(iii) determine the value of the v_θ at the same given near-earth-point such that the satellite orbit is a circular orbit , **(2 marks)**

(Hint : $E = \frac{1}{2}\mu v_\theta^2 - \frac{k}{r} \xrightarrow{\text{circular orbit}} -\frac{k}{2r}$)

(iv) determine the value of the v_θ at the same given near-earth-point such that the satellite orbit is a parabolic orbit . **(2 marks)**

Question four

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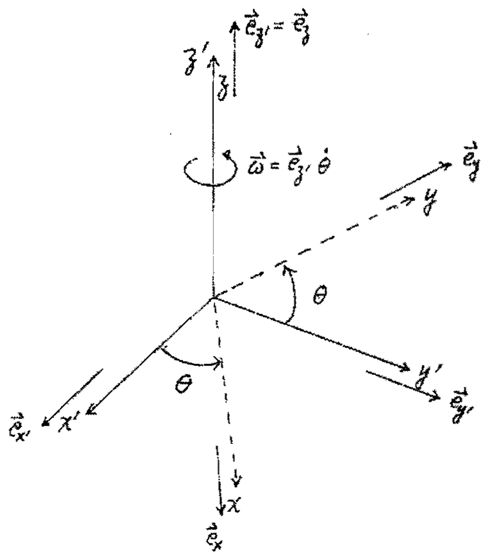
(a) Two set of Cartesian coordinate axes are having the same origins and z-axis. The non-prime system (referred to as "rotating" system) is rotating with an angular velocity $\vec{\omega} = \vec{e}_{z'}\,\dot{\theta}$ about the prime system (referred as "fixed" system) as shown below:



For any vector field $\vec{F}$ decomposed into the above two-set of cartesian components, i.e., $\vec{F} = \vec{e}_x\,F_x + \vec{e}_y\,F_y + \vec{e}_z\,F_z = \vec{e}_{x'}\,F_{x'} + \vec{e}_{y'}\,F_{y'} + \vec{e}_{z'}\,F_{z'}$, show that

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \left(\frac{d\vec{F}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{F}$$ where

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \vec{e}_{x'}\,\frac{dF_{x'}}{dt} + \vec{e}_{y'}\,\frac{dF_{y'}}{dt} + \vec{e}_{z'}\,\frac{dF_{z'}}{dt} \quad and$$

(12 marks)

$$\left(\frac{d\vec{F}}{dt}\right)_{rotating} = \vec{e}_x\,\frac{dF_x}{dt} + \vec{e}_y\,\frac{dF_y}{dt} + \vec{e}_z\,\frac{dF_z}{dt}$$

(Hint : $\vec{e}_x = \vec{e}_{x'}\,\cos(\theta) + \vec{e}_{y'}\,\sin(\theta)$, $\vec{e}_y = -\,\vec{e}_{x'}\,\sin(\theta) + \vec{e}_{y'}\,\cos(\theta)$ *and* $\vec{e}_z = \vec{e}_{z'}$)

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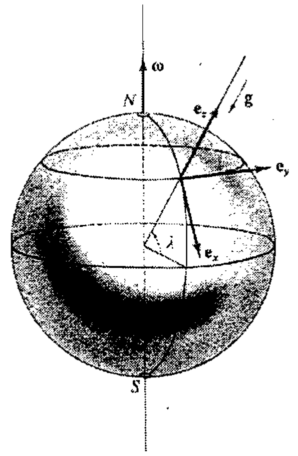
(b)



If a person, near the earth surface at a northern latitude λ , fired a bullet of speed, v_0 , at a target situated at his north direction ($-\vec{e}_x$ direction) of distance L away from him. Assuming he has a perfect rifle and the time T for the bullet hitting the target is short and $T \approx \frac{L}{v_0}$ (i.e., neglecting the gravitational bending and assuming the bullet is moving along $-x$ direction with constant speed v_0) .

(i) Show that the bullet will miss the target by a deviation distance d resulting from the Coriolis force $\left(-2\, m\, \vec{\omega} \times \vec{v}_r\right)$. Show that

$$d = \frac{\omega L^2}{v_0} \sin(\lambda)$$ **(11 marks)**

(Hint : $\vec{a}_{eff} \approx -2\, \vec{\omega} \times \vec{v}_r$, $\vec{v}_r \approx \vec{e}_x \left(-v_0\right)$, $\vec{\omega} = \vec{e}_x \left(-\omega \cos(\lambda)\right) + \vec{e}_z \left(\omega \sin(\lambda)\right)$)

(ii) Given the values of $\lambda = 60^0$, $L = 2000$ m , $v_0 = 800$ m/s and $\omega = 2\pi$ rad/day (i.e., $\omega = 7.27 \times 10^{-5}$ rad/s) , determine the value of the deviation distance d .

(2 marks)

Question five

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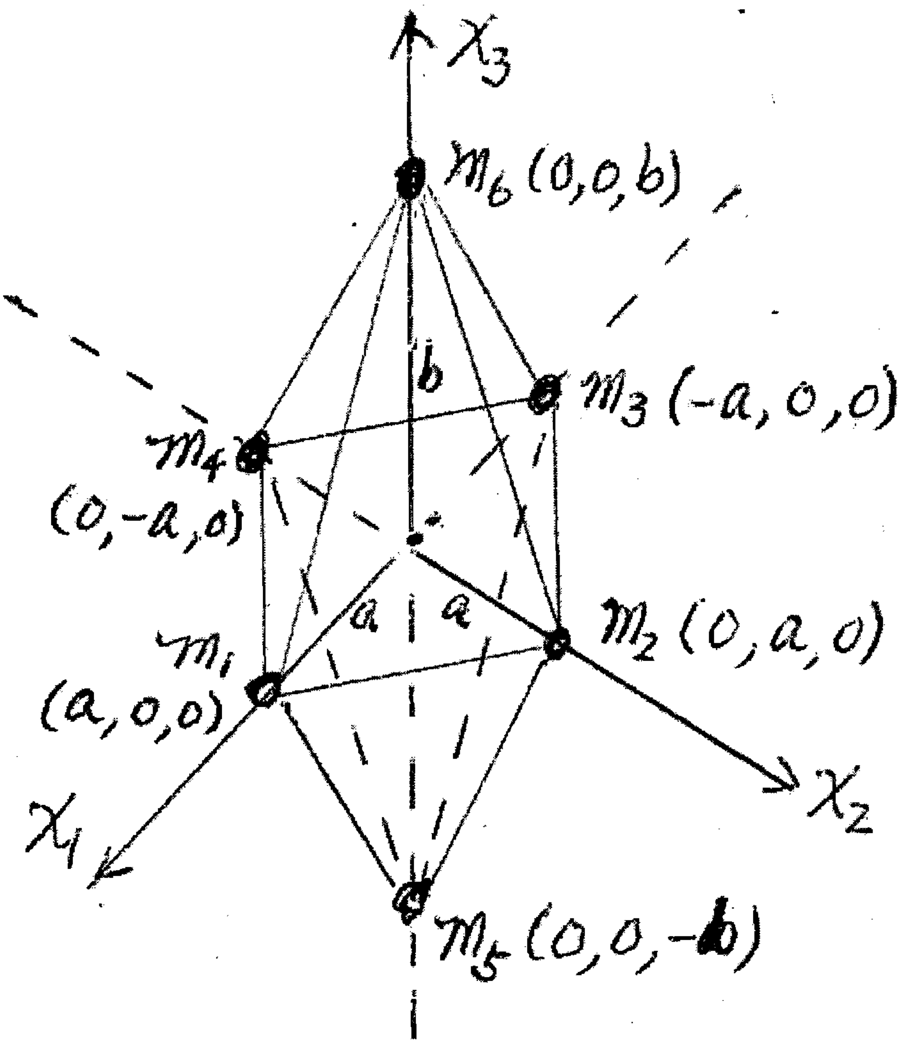
Six equal mass point $m\,(= m_1 = m_2 = \cdots = m_6)$ attached by massless rigid rods to form a rigid body of diamond shape with the center of mass of the "diamond" chosen as the origin of the body coordinate system (x_1, x_2, x_3) as shown in the diagram below.



where each mass point's coordinates in terms of length a & b is indicated in the diagram.

(a) Evaluate all elements of the inertia tensor, I, of the given rigid body with respect to the chosen body coordinate system and show that

$$I = \begin{pmatrix} 2\,m\left(a^2 + b^2\right) & 0 & 0 \\ 0 & 2\,m\left(a^2 + b^2\right) & 0 \\ 0 & 0 & 4\,m\,a^2 \end{pmatrix} .$$

(6 marks)

(b) If the given rigid body is only rotating with an angular velocity $\vec{\omega}$ without translational motion with respect to a fixed inertia coordinate system (x'_1, x'_2, x'_3) sharing the same origin as that of the body coordinate system, write down the total kinetic energy $T = T_{rotational} = \frac{1}{2}\vec{\omega} \bullet I \bullet \vec{\omega}$ in terms of $m, a, b, \omega_1, \omega_2 \& \omega_3$ where $\vec{\omega} = \vec{e}_1\,\omega_1 + \vec{e}_2\,\omega_2 + \vec{e}_3\,\omega_3$. **(2 marks)**

(c) The following are Euler's equations for force-free pure-rotational motion, i.e., $L = T_{rotational}$, for already diagonalized I as the case in (a).

$$\begin{cases} (I_2 - I_3)\,\omega_2\,\omega_3 - I_1\,\dot{\omega}_1 = 0 & \cdots\cdots\ (1) \\ (I_3 - I_1)\,\omega_3\,\omega_1 - I_2\,\dot{\omega}_2 = 0 & \cdots\cdots\ (2) \\ (I_1 - I_2)\,\omega_1\,\omega_2 - I_3\,\dot{\omega}_3 = 0 & \cdots\cdots\ (3) \end{cases}$$

(i) For our given rigid body, deduce from the above Euler's equations that

$$\begin{cases} \omega_3 = const. \xrightarrow{set\ as} K & \cdots\cdots\ (4) \\ \dot{\omega}_1 = \dfrac{(-a^2 + b^2)K}{(a^2 + b^2)}\,\omega_2 & \cdots\cdots\ (5) \\ \dot{\omega}_2 = -\dfrac{(-a^2 + b^2)K}{(a^2 + b^2)}\,\omega_1 & \cdots\cdots\ (6) \end{cases}$$

(6 marks)

(ii) If $b > a$ & $K > 0$, then $\dfrac{(-a^2 + b^2)K}{(a^2 + b^2)}$ is a positive constant and set it as Ω . Deduce from eq.(5) and eq.(6) in (c)(i) that $\ddot{\omega}_1 = -\Omega^2\,\omega_1$ $\cdots\cdots$ (7) **(3 marks)**

(iii) By direct substitution, show that $\omega_1 = A\cos(\Omega t + B)$ $\cdots\cdots$ (8) is the solution to eq.(7) with A & B constant values linking to the given initial value of $\vec{\omega}$. **(2 marks)**

(iv) Substitute eq.(8) into eq.(5) and deduce that $\omega_2 = -A\sin(\Omega t + B)$ $\cdots\cdots$ (9) **(2 marks)**

(v) Show that the magnitude of $\vec{\omega}$ is a constant for all time t. **(4 marks)**

Useful informations

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$$V = -\int \vec{F} \bullet d\vec{l} \quad \text{and reversely} \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \text{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \; \frac{N\, m^2}{kg^2}$$

$$\text{radius of earth} \quad r_E = 6.4 \times 10^6 \;\; m$$

$$\text{mass of earth} \quad m_E = 6 \times 10^{24} \;\; kg$$

$$\text{earth attractive potential} \equiv -\frac{k}{r} \quad \text{where} \quad k = G\, m\, m_E$$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k}} \quad \{(\varepsilon = 0, \text{circle}), (0 < \varepsilon < 1, \text{ellipse}), (\varepsilon = 1, \text{parabola}), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_2 >> m_1$$

$$\text{For elliptical orbit, i.e., } 0 < \varepsilon < 1, \text{ then} \begin{cases} \text{semi-major} \;\; a = \dfrac{k}{2\,|E|} \\ \text{semi-minor} \;\; b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ \text{period} \;\; \tau = \dfrac{2\,\mu}{l} (\pi\, a\, b) \\ r_{min} = a\,(1 - \varepsilon) \;\; \& \; r_{max} = a\,(1 + \varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r \left(\ddot{r} - r\, \dot{\theta}^2 \right) + \vec{e}_\theta \left(2\, \dot{r}\, \dot{\theta} + r\, \ddot{\theta} \right) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

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$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,2} & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,1} & -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$