UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2016/2017

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS TEN PAGES, INCLUDING THIS PAGE.

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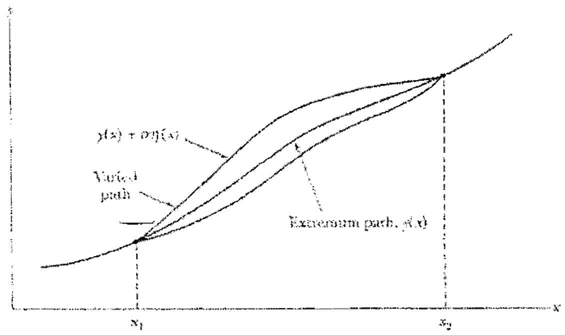
Question one

(a) Given the following definite integral of $J(\alpha) = \int_{x_1}^{x_2} f(y(\alpha, x), y'(\alpha, x), y''(\alpha, x); x)\, dx$, where the varied integration path is $y(\alpha, x) = y(x) + \alpha\, \eta(x)$, $\eta(x_1) = \eta(x_2) = 0$ *and* $\left.\frac{d\,\eta(x)}{d\,x}\right|_{x = x_1} = \left.\frac{d\,\eta(x)}{d\,x}\right|_{x = x_2} = 0$ as shown in the following diagram :



Use the extremum condition for $J(\alpha)$, i.e., $\left.\frac{\partial J(\alpha)}{\partial \alpha}\right|_{\alpha = 0} = 0$, to deduce that f *along the extremum path , i.e.,* $f(y(x), y'(x), y''(x); x)$, satisfies the following equation:

$$\frac{\partial f}{\partial y} - \frac{d}{d\,x}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{d\,x^2}\left(\frac{\partial f}{\partial y''}\right) = 0 \; .$$ **(12 marks)**

(b) For a certain dynamical system the kinetic energy, T, and potential energy, V, are given by

$$T = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_1\, \dot{q}_2 + \dot{q}_2^2\right)$$

$$V = \frac{3}{2}\, q_2^2$$

where q_1 , q_2 are the generalized coordinates.

(i) Write down Lagrange's equations of motion. **(5 marks)**

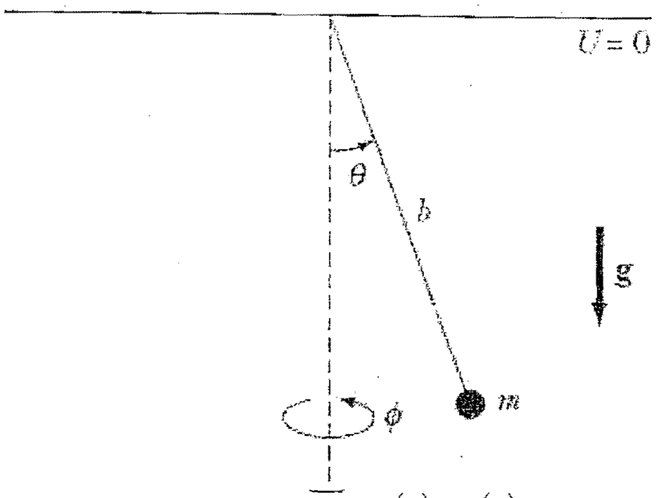
(ii) From the results in (b)(i), deduce a differential equation for q_2 only and show that its general solution can be written as $q_2(t) = k_1 \cos(2t) + k_2 \sin(2t)$ where k_1 *and* k_2 are arbitrary constants . **(4 marks)**

(iii) Substitute the general solution of q_2 into the results in (b)(i) and show that the the general solution of q_1 can be written as
$q_1(t) = 2\,k_1 \cos(2t) + 2\,k_2 \sin(2t) + k_3\, t + k_4$
where k_3 *and* k_4 are arbitrary constants . **(4 marks)**

Question two

95

A spherical pendulum of mass m and length b is shown in the figure below:



(a) (i) From $x = b\sin(\theta)\cos(\phi)$, $y = b\sin(\theta)\sin(\phi)$ & $z = -b\cos(\theta)$ and $T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ & $V = m\,g\,z$, deduce the following Lagrangian for the system in terms of θ & ϕ as

$$L = \frac{1}{2} m\,b^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2(\theta)\right) + m\,g\,b\cos(\theta) \quad \cdots\cdots \quad (1)$$

(5 marks)

(ii) Write down the equations of motion for θ & ϕ and deduce that

$$\begin{cases} \dfrac{d\,p_\theta}{d\,t} = m\,b^2\sin(\theta)\cos(\theta)\,\dot{\phi}^2 - m\,g\,b\sin(\theta) & \cdots\cdots \quad (2) \\ \dfrac{d\,p_\phi}{d\,t} = 0 & \cdots\cdots \quad (3) \end{cases}$$

$$where \quad p_\theta = m\,b^2\,\dot{\theta} \quad \& \quad p_\phi = m\,b^2\sin^2(\theta)\,\dot{\phi}$$

(5 marks)

(iii) From eq.(3), one has $p_\phi = const. \xrightarrow{set\ as} K$, deduce from eq.(2) the following equation for small θ , i.e., $\left(\sin(\theta) \approx \theta \;\; and \;\; \cos(\theta) \approx 1 - \frac{\theta^2}{2} \;\; or \;\; 1\right)$, that

$$m^2\,b^4\,\theta^3\,\ddot{\theta} = K^2 - m^2\,g\,b^3\,\theta^4 \quad \cdots\cdots \quad (4)$$

(4 marks)

(iv) If $K = 0$ in eq.(4), write down the general solution of $\theta(t)$. **(3 marks)**

(b) (i) Find the Hamiltonian of the system in terms of θ , ϕ , p_θ & p_ϕ .

(Hint : Since the Lagrangian in (a)(i) is not explicitly depending on t , thus the Hamiltonian H equals to $T + V$) **(4 marks)**

(ii) Write down the equations of motion for H in (b)(i). **(4 marks)**

Question three

96

(a) Given the Lagrangian for the two-body central force system as :

$$L = T - V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + \frac{k}{r^n}$$

where μ is the reduced mass of the system, k & n are positive constants and (r,θ) are polar coordinates of the motion plane with its origin at the center of mass of the two-body system.

(i) Write down the Lagrange's equations of motion for the given system and show that

$$\begin{cases} \dot{\theta} = \dfrac{l}{\mu\, r^2} & \cdots\cdots\ (1) \\[2ex] \mu\,\ddot{r} - \dfrac{l^2}{\mu\, r^3} + \dfrac{n\,k}{r^{n+1}} = 0 & \cdots\cdots\ (2) \end{cases}$$

where the angular momentum l is a constant. **(5 marks)**

(ii) In the case of circular orbits, i.e., $r = const.$, use the results in (a)(i) and $T = \frac{1}{2}\mu\left(\cancel{\dot{r}^2} + r^2\,\dot{\theta}^2\right)$ & $V = \frac{k}{r^n}$ to find a relation between the kinetic and potential energies and show that $T = -\frac{n}{2}V$ **(5 marks)**

(b) Starting from the law of conservation of angular momentum $l\ \left(= \mu\, r^2\,\dot{\theta}\right)$, derive Kepler's third law , i.e., the relation between the period τ of a closed orbit and the area A of the closed orbit. Show that $\tau = \frac{2\,\mu}{l}A$ **(6 marks)**

(c) If an earth satellite of 400 kg mass is having a pure tangential speed $v_\theta\ \left(= r\,\dot{\theta}\right) = 10{,}000 = 10^4$ m/s at its near-earth-point 500 km above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(3 marks)**

(ii) calculate the values of the eccentricity, ε , and show that the orbit is an elliptical orbit. Also calculate its period. **(6 marks)**

Question four

(a) Two set of coordinate systems are having the same origins. The non-prime system (with position vector denoted as $\vec{r}$ and referred to as "rotating" system) is rotating with an angular velocity $\vec{\omega}$ about the prime system (with position vector denoted as $\vec{r}\,'$ and referred as "fixed" system and taken as an inertial system). Use the relation that

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \left(\frac{d\vec{F}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{F} \quad \text{for any vector field } \vec{F} \text{ where}$$

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} \ \& \ \left(\frac{d\vec{F}}{dt}\right)_{rotating}$$ are time derivatives of $\vec{F}$ with respect to the fixed and rotating systems respectively, deduce the following:

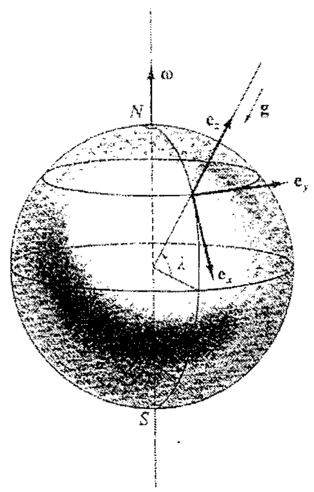
$$\vec{a}_{eff} = \vec{a} - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2\,\vec{\omega}\times\vec{v}_r \qquad \text{where}$$

$$\vec{v}_f \equiv \left(\frac{d\vec{r}\,'}{dt}\right)_{fixed}, \ \vec{v}_r \equiv \left(\frac{d\vec{r}}{dt}\right)_{rotating}, \ \vec{a} \equiv \left(\frac{d\vec{v}_f}{dt}\right)_{fixed}, \ \vec{a}_{eff} \equiv \left(\frac{d\vec{v}_r}{dt}\right)_{rotating}$$

$$and \quad \dot{\vec{\omega}} \equiv \left(\frac{d\vec{\omega}}{dt}\right)_{rotating}$$

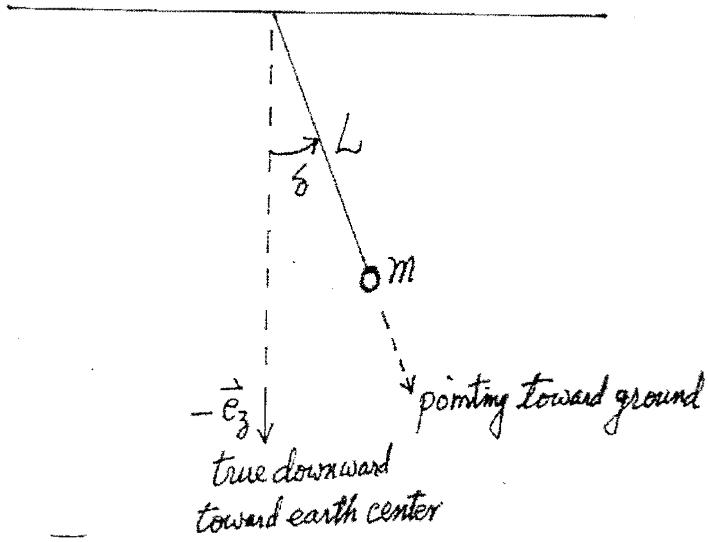
(Hint : Starting with the equation $\vec{r}\,' = \vec{r}$ (same origin) and taking the time derivative of the equation twice with respect to the fixed system.) **(10 marks)**

(b)



Referring to the diagram above and considering the body coordinate system (x, y, z) has the same origin as the earth's fixed inertial system (x', y', z') , i.e., center of the earth . Hanging a motionless simple pendulum of length L and mass m near the earth surface at a northern latitude λ , the pendulum is supposed to pointing direct downward along $-\vec{e}_z$ *direction* but instead it is pointing toward the ground with a small angular deviation of δ made with the true downward direction resulting from the centrifugal force $(-m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}))$ as shown in the diagram below.

Question four (continued)



(i) Show that $\delta \approx \dfrac{\omega^2\, r_E \cos(\lambda)\sin(\lambda)}{g - \omega^2\, r_E \cos^2(\lambda)}$. **(12 marks)**

(Hint : $\vec{F}_{eff} \approx \vec{e}_z\,(-m\,g) - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$, $\vec{r} \approx \vec{e}_z\,(r_E)$ &

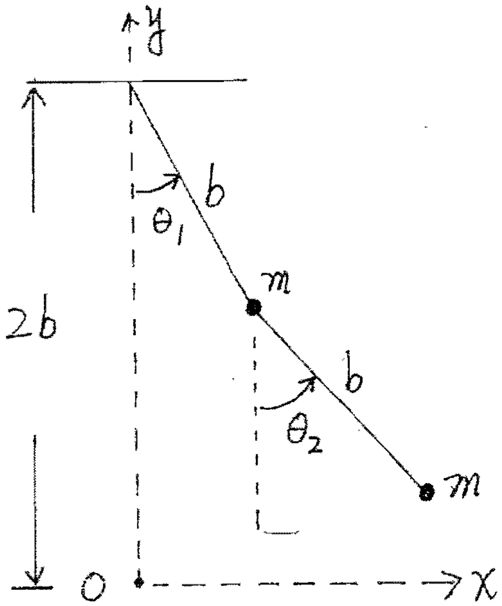
$\vec{\omega} = \vec{e}_x\,(-\omega\cos(\lambda)) + \vec{e}_z\,(\omega\sin(\lambda))$)

(ii) Given $\lambda = 30^0$, $L = 10$ m , $r_E = 6400$ km , $g = 9.8$ m/s^2 and $\omega = 2\pi$ rad/day $= 7.27\times10^{-5}$ rad/s , determine the value of the deviation angle δ in terms of degree. **(3 marks)**

Question five

99

Consider the motion of the bobs in the double pendulum system in the figure below.



Both pendulums are identical and having the length b and bob of mass m . The motion of both bobs is restricted to lie in the plane of this paper , i.e., x-y plane.

(i) For small θ_1 *and* θ_2 , i.e., $\left(\sin(\theta) \approx \theta \quad and \quad \cos(\theta) \approx 1 - \dfrac{\theta^2}{2} \quad or \quad 1 \right)$, show that the Lagrangian for the system can be expressed as:

$$L = m\, b^2\, \dot{\theta}_1^2 + \frac{1}{2}\, m\, b^2\, \dot{\theta}_2^2 + m\, b^2\, \dot{\theta}_1\, \dot{\theta}_2 - m\, g\, b \left(1 + \theta_1^2 + \frac{\theta_2^2}{2} \right)$$

where the zero gravitational potential is set at the equilibrium position of the lower bob , i.e., $\theta_1 = 0$, $\theta_2 = 0$ *and* $y = 0$. **(6 marks)**

(ii) Write down the equations of motion and deduce that

$$\begin{cases} 2\,\ddot{\theta}_1 + \ddot{\theta}_2 = -2\,\dfrac{g}{b}\,\theta_1 \\ \ddot{\theta}_1 + \ddot{\theta}_2 = -\dfrac{g}{b}\,\theta_2 \end{cases}$$

(4 marks)

(iii) Deduce from the equations in (ii) the following :

$$\begin{cases} \ddot{\theta}_1 = -2\,\dfrac{g}{b}\,\theta_1 + \dfrac{g}{b}\,\theta_2 \\ \ddot{\theta}_2 = 2\,\dfrac{g}{b}\,\theta_1 - 2\,\dfrac{g}{b}\,\theta_2 \end{cases}$$

(2 marks)

(iv) Set $\theta_1 = \hat{X}_1\, e^{i\omega t}$ *and* $\theta_2 = \hat{X}_2\, e^{i\omega t}$ (where $\hat{X}_1$ *and* $\hat{X}_2$ are constants) and deduce from the equations in (iii) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(2\dfrac{g}{b}\right) & \dfrac{g}{b} \\ 2\dfrac{g}{b} & -\left(2\dfrac{g}{b}\right) \end{pmatrix}$$

(3 marks)

(v) Find the eigenfrequencies, ω , of this coupled system. **(5 marks)**

(vi) Find the eigenvectors of this coupled system. **(5 marks)**

Useful informations

'01

$$V = - \int \vec{F} \bullet d\vec{l} \quad \textit{and reversely} \quad \vec{F} = - \vec{\nabla} V$$

$$L = T - V = L(q_1 , q_2 , \cdots , q_n , \dot{q}_1 , \dot{q}_2 , \cdots , \dot{q}_n , t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \textit{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha = 1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1 , q_2 , \cdots , q_n , \dot{q}_1 , \dot{q}_2 , \cdots , \dot{q}_n , t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \textit{and} \quad \dot{p}_\alpha = - \frac{\partial H}{\partial q_\alpha}$$

$$[u , v] \equiv \sum_{\alpha = 1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N \, m^2}{kg^2}$$

radius of earth $r_E = 6.4 \times 10^6 \ m$

mass of earth $m_E = 6 \times 10^{24} \ kg$

earth attractive potential $\equiv - \dfrac{k}{r}$ *where* $k = G \, m \, m_E$

$$\varepsilon = \sqrt{1 + \frac{2 \, E \, l^2}{\mu \, k}} \quad \{(\varepsilon = 0 \, , \textit{circle}), (0 < \varepsilon < 1 \, , \textit{ellipse}), (\varepsilon = 1 \, , \textit{parabola}), \cdots\}$$

$$\mu = \frac{m_1 \, m_2}{m_1 + m_2} \approx m_1 \quad \textit{if} \quad m_2 >> m_1$$

$$\textit{For elliptical orbit, i.e., } 0 < \varepsilon < 1 \textit{, then} \begin{cases} \textit{semi-major} \ \ a = \dfrac{k}{2 \, |E|} \\ \textit{semi-minor} \ \ b = \dfrac{l}{\sqrt{2 \, \mu \, |E|}} \\ \textit{period} \ \ \tau = \dfrac{2 \, \mu}{l} (\pi \, a \, b) \\ r_{\min} = a \, (1 - \varepsilon) \ \ \& \ r_{\max} = a \, (1 + \varepsilon) \end{cases}$$

for plane polar (r , θ) *system with unit vectors* $(\vec{e}_r , \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r \, \dot{r} + \vec{e}_\theta \, r \, \dot{\theta} \\ \vec{a} = \vec{e}_r \left(\ddot{r} - r \, \dot{\theta}^2 \right) + \vec{e}_\theta \left(2 \, \dot{r} \, \dot{\theta} + r \, \ddot{\theta} \right) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

102

$$I = \begin{pmatrix} \sum_{\alpha} m_{\alpha} \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_{\alpha} m_{\alpha}\, x_{\alpha,1}\, x_{\alpha,2} & -\sum_{\alpha} m_{\alpha}\, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha}\, x_{\alpha,2}\, x_{\alpha,1} & \sum_{\alpha} m_{\alpha} \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_{\alpha} m_{\alpha}\, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha}\, x_{\alpha,3}\, x_{\alpha,1} & -\sum_{\alpha} m_{\alpha}\, x_{\alpha,3}\, x_{\alpha,2} & \sum_{\alpha} m_{\alpha} \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$