UNIVERSITY OF SWAZILAND

/03

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2016/2017

TITLE OF PAPER : ELECTROMAGNETIC THEORY

COURSE NUMBER : P331

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS TEN PAGES, INCLUDING THIS PAGE.

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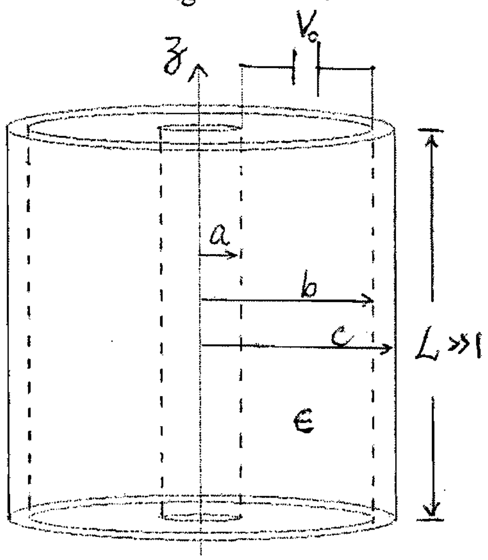
Question one

A very long straight coaxial cable, with an inner solid wire of radius a and an outer hollow wire with inner radius b and outer radius c, is given a potential difference V_0 across the wires. In between the wires $(a < \rho < b)$ is filled with a layer of insulating material with permittivity ε as shown in the figure below.



(a) From $\nabla^2 f(\rho) = 0$ with boundary conditions of $f(\rho = a) = 0$ and $f(\rho = b) = V_0$, find the specific solution of $f(\rho)$ and show that

$$f(\rho) = \frac{V_0}{\ln\left(\frac{b}{a}\right)} \ln\left(\frac{\rho}{a}\right) .$$

(9 marks)

(b) Find the electric field $\vec{E}$ from $f(\rho)$ obtained in (a). **(3 marks)**

(c) (i) Find the surface conduction charge density ρ_s on $\rho = a$ and $\rho = b$ conducting surfaces respectively. Then find the total charges deposited on both surfaces, if the total cable length is L, and show that they are equal and opposite. **(4+4 marks)**

(Hint : For the conducting surfaces in contact with the dielectric region of ε, then $\rho_s = \vec{e}_n \bullet (\varepsilon \vec{E})$ where $\vec{e}_n$ is the normal outward unit vector on conductor surface.)

(ii) Write down the capacitance C as well as the distributive capacitance c_d of the given coaxial cable. Show that

$$c_d = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} .$$

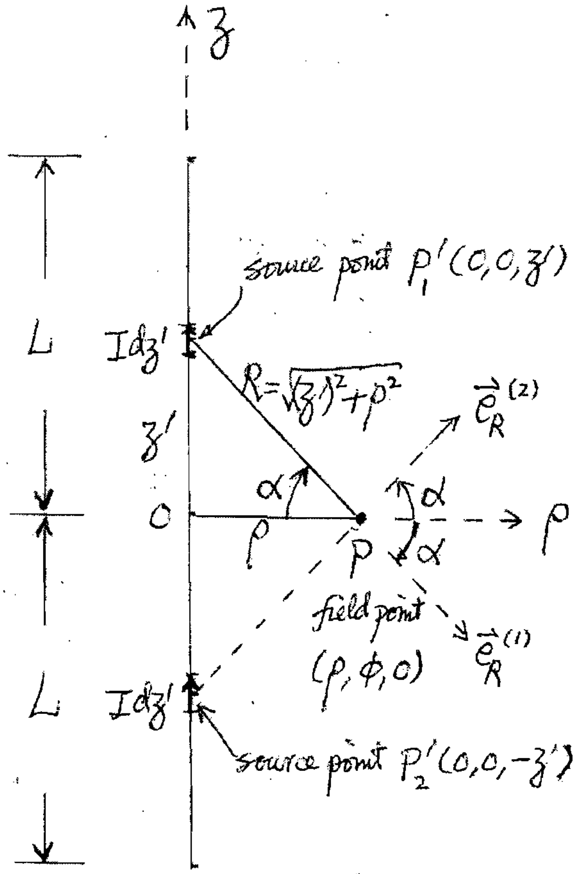
(2 marks)

(iii) If given the values of $a = 2$ mm, $b = 8$ mm & $\varepsilon = 3\varepsilon_0$, calculate the value of c_d. **(3 marks)**

Question two

/05

(a) A thin conducting wire of length $2L$, with its central axis coinciding with the z-axis and its centre point coinciding with the origin, carries a steady total current I along positive z-direction as shown in the figure below.



(i) Since the given current source is only along z-axis the vector potential at $P(\rho,\phi,0)$ is also having only z component A_z, i.e.,

$$\vec{A} = \vec{e}_z A_z \quad where \quad A_z = \int_{z'=-L}^{z'=+L} \frac{\mu_0 I\, dz'}{4\pi R} = 2\int_{z'=0}^{z'=+L} \frac{\mu_0 I\, dz'}{4\pi\sqrt{(z')^2+\rho^2}},$$

carry out the above integral for A_z about z' and show that

$$A_z = \frac{\mu_0 I}{2\pi}\ln\left(\frac{L+\sqrt{L^2+\rho^2}}{\rho}\right)$$ **(8 marks)**

(Hint : set $z' = \rho\tan(\alpha)$, $\int \sec(\alpha)\, d\alpha = \ln(\sec(\alpha)+\tan(\alpha))$)

(ii) For $L >> \rho$ use $\vec{A} \approx \vec{e}_z \frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{\rho}\right)$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ to show that

$$\vec{B} = \vec{e}_\phi \frac{\mu_0 I}{2\pi\rho}.$$ **(5 marks)**

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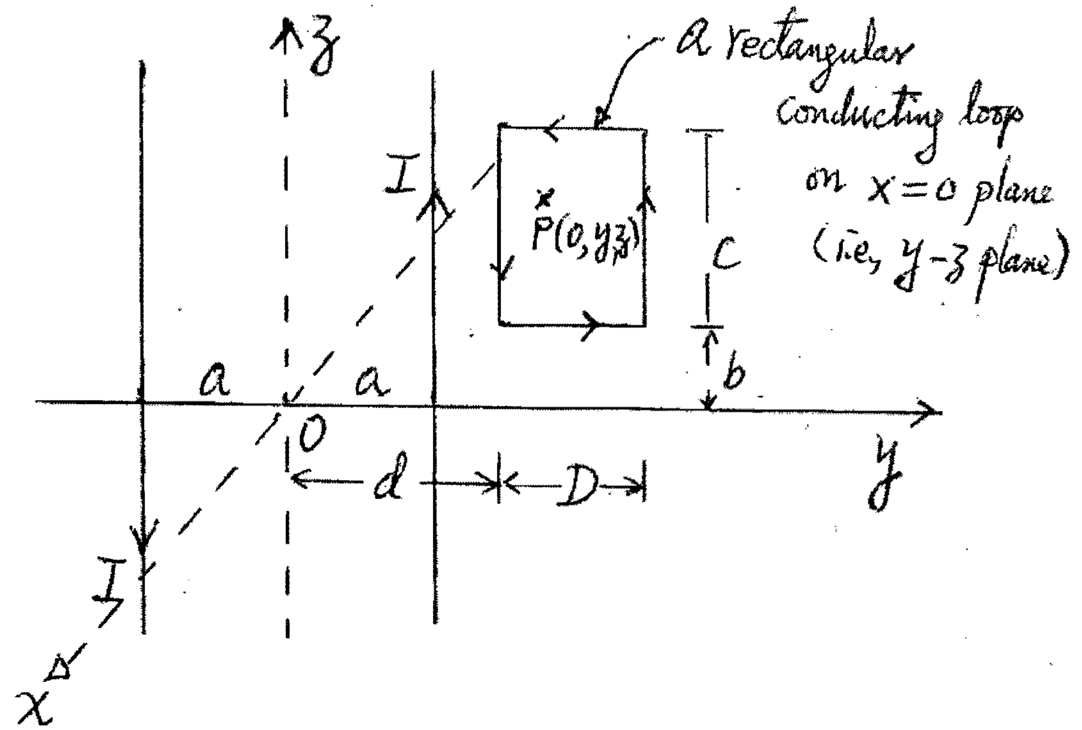
(b) Two very long thin conducting wires parallel to z-axis and lying on the $x = 0$ plane, i.e, $y - z$ *plane*, one situated at $y = -a$ and carries a current I A along - z direction and the other situated at $y = +a$ and carries a current I A along + z direction as shown in the following diagram.



A rectangular conducting loop of dimension $D \times c$ is placed on $x = 0$ plane and a distance of d away from the z-axis as shown in the above diagram.

(i) Utilize the result in (a)(ii), apply the superposition principle to deduce that the magnetic field at point $P:(0,y,z)$ within the rectangular loop due to the two parallel conducting wires is

$$\vec{B}(0,y,z) = \vec{e}_x \frac{\mu_0 I}{2\pi}\left(\frac{1}{y+a} - \frac{1}{y-a}\right)$$ **(4 marks)**

(Hint: on $x = 0$ plane $\vec{e}_\phi = -\vec{e}_x$)

(ii) Find the total magnetic flux $\Phi_m = \int_S \vec{B} \bullet d\vec{s}$ passing through the surface area confined by the rectangular loop, i.e., $S:\ x = 0\ ,\ d \le y \le d + D\ ,\ b \le z \le b + c$ and $d\vec{s} = \vec{e}_x\, dy\, dz$, in terms of μ_0, a, b, c, d, D & I. Show that the mutual inductance between the rectangular loop and the parallel wires is

$$M = \frac{\mu_0\, c}{2\pi} \ln\left(\frac{(d+D+a)(d-a)}{(d+D-a)(d+a)}\right)$$ **(8 marks)**

Question three 107

(a) Apply an electric field $\vec{E}$ to a pure conducting solid material. According to modified Drude's model, the equation of motion for an average conduction electron in the solid material can be written as

$$m_e \frac{d\vec{v}_d}{dt} = -e\,\vec{E} - \frac{2\,m_e\,\vec{v}_d}{\tau_f}$$

where $-e$ & m_e are the electron charge and mass respectively.

(i) Explain briefly the meanings of $\vec{v}_d$, τ_f and $-\dfrac{2\,m_e\,\vec{v}_d}{\tau_f}$. **(3 marks)**

(ii) In the steady state case, i.e., $\dfrac{d\vec{v}_d}{dt} = 0$, deduce the following point form of Ohm's law $\vec{J} = \sigma\,\vec{E}$ *where* $\sigma = \dfrac{n\,e^2}{2\,m_e}\tau_f$ and n is the number density of the conduction electrons in the material. **(6 marks)**
(Hint : $\vec{J} = \rho_v\,\vec{v}_d = -n\,e\,\vec{v}_d$)

(iii) Pure solid Ruthenium Ru has an atomic weight = 101.07 kg / kg-mole, a density = 12200 kg/m^3 and a conductivity $\sigma = 1.4 \times 10^7\ \Omega^{-1}\,m^{-1}$ at room temperature.

(A) Find the number of conduction electrons per meter cube, i.e., number density n , for metal Ru if each Ru atom contributes two conduction electrons. **(4 marks)**
(Hint : one kg–mole pure metal contains 6.022×10^{26} atoms)

(B) Find the value of τ_f for Ru metal at room temperature. **(4 marks)**

(b) The Maxwell's equations for a material region with parameters of $(\mu\,,\varepsilon\,,\sigma)$ are

$$\begin{cases} \vec{\nabla} \bullet \vec{E}(space,t) = 0 & \cdots\cdots\ (1) \\ \vec{\nabla} \bullet \vec{H}(space,t) = 0 & \cdots\cdots\ (2) \\ \vec{\nabla} \times \vec{E}(space,t) = -\,\mu\,\dfrac{\partial \vec{H}(space,t)}{\partial t} & \cdots\cdots\ (3) \\ \vec{\nabla} \times \vec{H}(space,t) = \sigma\,\vec{E}(space,t) + \varepsilon\,\dfrac{\partial \vec{E}(space,t)}{\partial t} & \cdots\cdots\ (4) \end{cases}$$

(i) Setting $\vec{E}(space,t) = \tilde{\vec{E}}(space)\,e^{i\omega t}$ & $\vec{H}(space,t) = \tilde{\vec{H}}(space)\,e^{i\omega t}$, deduce the following time-harmonic Maxwell's equations :

$$\begin{cases} \vec{\nabla} \bullet \tilde{\vec{E}}(space) = 0 & \cdots\cdots\ (5) \\ \vec{\nabla} \bullet \tilde{\vec{H}}(space) = 0 & \cdots\cdots\ (6) \\ \vec{\nabla} \times \tilde{\vec{E}}(space) = -\,i\,\omega\,\mu\,\tilde{\vec{H}}(space) & \cdots\cdots\ (7) \\ \vec{\nabla} \times \tilde{\vec{H}}(space) = (\sigma + i\,\omega\,\varepsilon)\,\tilde{\vec{E}}(space) & \cdots\cdots\ (8) \end{cases}$$

(3 marks)

(ii) From equations in (b)(i) deduce the following wave equation for $\tilde{\vec{H}}(space)$ as

$\nabla^2\,\tilde{\vec{H}}(space) = \hat{\gamma}^2\,\tilde{\vec{H}}(space)$ *where* $\hat{\gamma} = \sqrt{i\,\omega\,\mu(\sigma + i\,\omega\,\varepsilon)}$ **(5 marks)**

Question four

/08

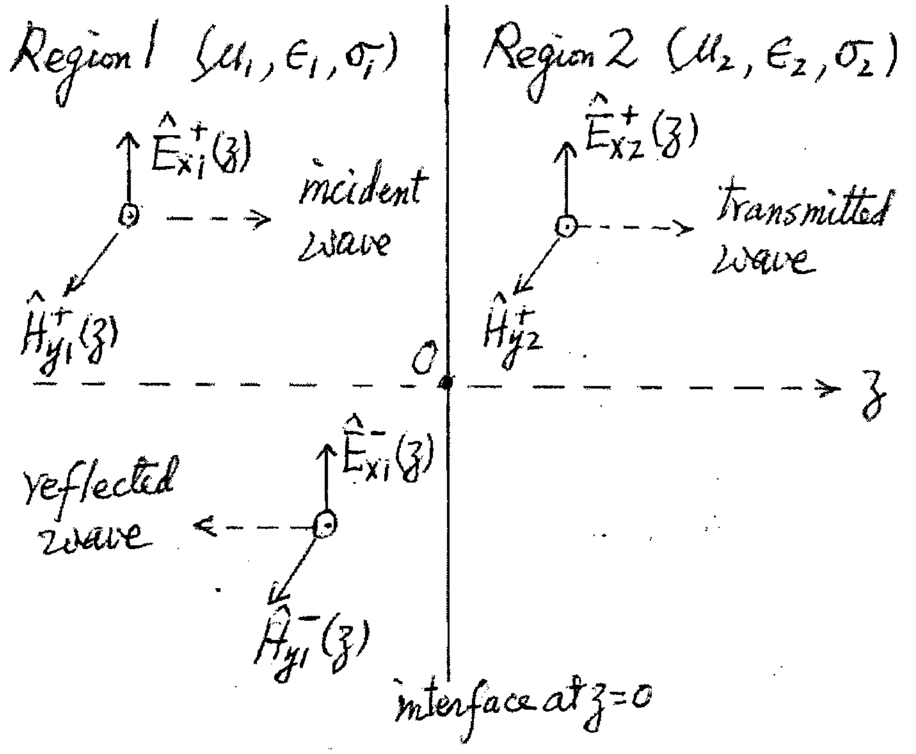
(a) A uniform plane wave traveling along $+z$ direction with the field components $E_x(z)$ & $H_y(z)$ has a complex electric field amplitude $\hat{E}_m^+ = 80\, e^{i\,40^0}$ V/m and propagates at a frequency $f = 6\times 10^5$ Hz in a material region has the parameters of $\mu = \mu_0$, $\varepsilon = 4\,\varepsilon_0$ & $\dfrac{\sigma}{\omega\,\varepsilon} = 0.7$.

(i) Find the values of the propagation constant $\hat{\gamma}\ (= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave. **(4 marks)**

(ii) Express the electric and magnetic fields in both their complex and real-time forms, with the numerical values of (a)(i) inserted. **(4 marks)**

(iii) Find the values of the penetration depth, wave length and phase velocity of the given wave. **(3 marks)**

(b) An uniform plane wave is incident normally upon an interface separating two regions . The incident wave is given as $\left(\hat{E}_{x1}^{+} = \hat{E}_{m1}^{+}\, e^{-\hat{\gamma}_1 z} \,,\, \hat{H}_{y1}^{+} = \dfrac{\hat{E}_{m1}^{+}}{\hat{\eta}_1}\, e^{-\hat{\gamma}_1 z} \right)$ and thus the reflected and transmitted wave can be written as $\left(\hat{E}_{x1}^{-} = \hat{E}_{m1}^{-}\, e^{+\hat{\gamma}_1 z} \,,\, \hat{H}_{y1}^{-} = -\dfrac{\hat{E}_{m1}^{-}}{\hat{\eta}_1}\, e^{+\hat{\gamma}_1 z} \right)$ and $\left(\hat{E}_{x2}^{+} = \hat{E}_{m2}^{+}\, e^{-\hat{\gamma}_2 z} \,,\, \hat{H}_{y2}^{+} = \dfrac{\hat{E}_{m2}^{+}}{\hat{\eta}_2}\, e^{-\hat{\gamma}_2 z} \right)$ respectively as shown below:



Question four (continued)

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(i) From the boundary conditions at the interface , i.e., both total $\hat{E}_x$ & $\hat{H}_y$ are continuous at $z = 0$, deduce the following

$$\begin{cases} \hat{E}^{-}_{m1} = \hat{E}^{+}_{m1} \dfrac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \\ \hat{E}^{+}_{m2} = \hat{E}^{+}_{m1} \dfrac{2\,\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_1} \end{cases}$$

(8 marks)

(ii) If region1 is air (i.e., $\hat{\eta}_1 = 120\,\pi = 377 \quad \Omega$) , region 2 is a lossy medium with parameters of $\left(\mu_2 = \mu_0 \ , \varepsilon_2 = 9\,\varepsilon_0 \ , \dfrac{\sigma_2}{\omega\,\varepsilon_2} = 1 \right)$, and the incident plane wave is having a complex amplitude of $\hat{E}^{+}_{m1} = 100\, e^{i\,50^{0}}$ V/m and propagates at a frequency of $f = 10^6$ Hz .

(A) Calculate the value of $\hat{\eta}_2$. **(2 marks)**

(B) Calculate the values of $\hat{E}^{-}_{m1}$. **(4 marks)**

Question five

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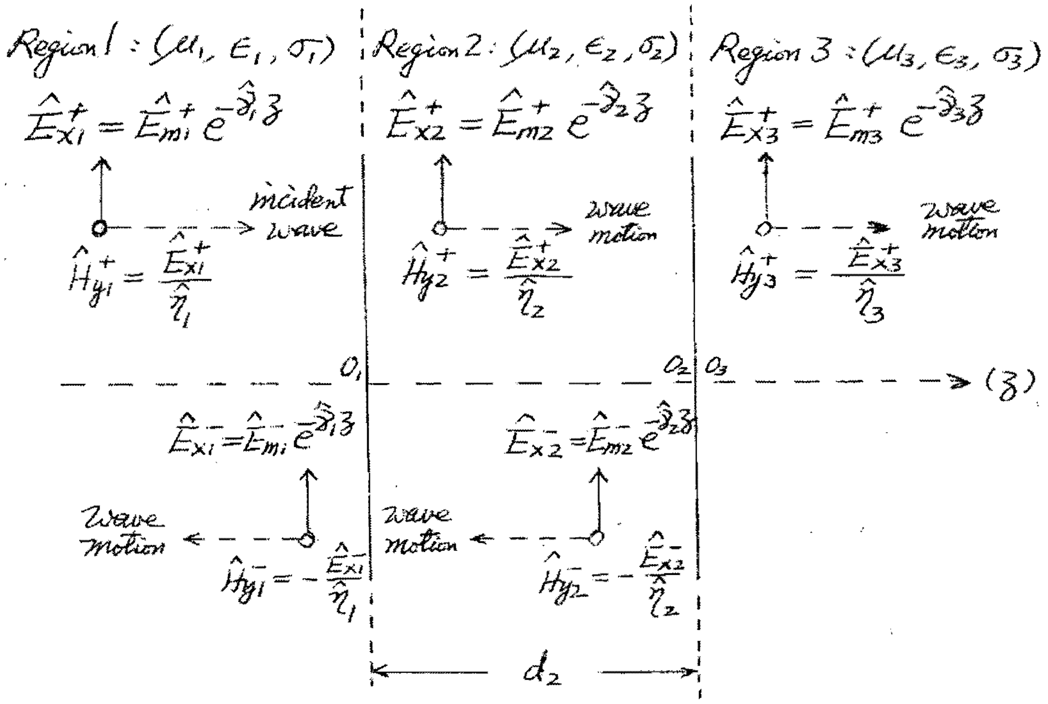
A uniform plane wave $\left(\hat{E}_{x1}^{+}, \hat{H}_{y1}^{+}\right)$, with a frequency f, is incident normally on a layer of thickness d_2, and emerges into region *3* as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface .

(a) Define for the i^{th} *region* $(i = 1,2,3)$ the reflection coefficient $\hat{\Gamma}_i(z)$ and the total wave impedance $\hat{Z}_i(z)$ and deduce the following :

$$\begin{cases} \hat{Z}_i(z) = \hat{\eta}_i \dfrac{1+\hat{\Gamma}_i(z)}{1-\hat{\Gamma}_i(z)} \\ \hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2\hat{\gamma}_i (z'-z)} \quad \textit{where } z' \ \& \ z \textit{ are two positions in } i^{th} \textit{ region} \end{cases}$$

(2 + 7 marks)

(b) If $f = 10^7$ Hz and $d_2 = \dfrac{\lambda_2}{4}$, region 1 & 3 are air regions and region 2 is a lossless region with parameters $\mu_2 = \mu_0$, $\varepsilon_1 = 16\,\varepsilon_0$ & $\dfrac{\sigma}{\omega\,\varepsilon} = 0$,

(i) find the values of β_1 , β_2 , β_3 , λ_2 & $\hat{\eta}_2$, (note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\,\pi\ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(4 marks)**

(ii) starting with $\hat{\Gamma}_3(z) = 0$ for the rightmost region , i.e., region 3 , and using continuous $\hat{Z}$ at the interface as well as the equations in (a) , find the values of $\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-d_2)$, $\hat{Z}_2(-d_2)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$ **(10 marks)**

(iii) find the value of $\hat{E}_{m1}^{-}$ *if given* $\hat{E}_{m1}^{+} = 50$ V/m . **(2 marks)**

Useful informations

$e = 1.6 \times 10^{-19}\ C$

$m_e = 9.1 \times 10^{-31}\ kg$

$\mu_0 = 4\,\pi \times 10^{-7}\ \dfrac{H}{m}$

$\varepsilon_0 = 8.85 \times 10^{-12}\ \dfrac{F}{m}$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3 \times 10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{\,i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\,\pi\ \ \Omega = 377\ \ \Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B}\bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\oint_L \vec{B}\bullet d\vec{l} = \mu\iint_S \vec{J}\bullet d\vec{s} + \mu\,\varepsilon\,\frac{\partial}{\partial t}\left(\iint_S \vec{E}\bullet d\vec{s}\right)$$

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla}\bullet\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\,\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

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$$\vec{D} = \varepsilon\,\vec{E} = \varepsilon_0\,\vec{E} + \vec{P} \qquad \& \qquad \vec{B} = \mu\,\vec{H} = \mu_0\,\vec{H} + \vec{M}$$

$$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla}\, f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2\,\vec{F}$$

$$\vec{\nabla}\, f = \vec{e}_x\,\frac{\partial f}{\partial x} + \vec{e}_y\,\frac{\partial f}{\partial y} + \vec{e}_z\,\frac{\partial f}{\partial z} = \vec{e}_\rho\,\frac{\partial f}{\partial \rho} + \vec{e}_\phi\,\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z\,\frac{\partial f}{\partial z}$$

$$= \vec{e}_r\,\frac{\partial f}{\partial r} + \vec{e}_\theta\,\frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi\,\frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\,\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\,r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x\left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y\left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z\left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\,\rho)}{\partial z}\right) + \vec{e}_\phi\left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\,\rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\,r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\,r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\,r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\,r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x\,F_x + \vec{e}_y\,F_y + \vec{e}_z\,F_z = \vec{e}_\rho\,F_\rho + \vec{e}_\phi\,F_\phi + \vec{e}_z\,F_z = \vec{e}_r\,F_r + \vec{e}_\theta\,F_\theta + \vec{e}_\phi\,F_\phi$ and

$$d\vec{l} = \vec{e}_x\,dx + \vec{e}_y\,dy + \vec{e}_z\,dz = \vec{e}_\rho\,d\rho + \vec{e}_\phi\,\rho\,d\phi + \vec{e}_z\,dz = \vec{e}_r\,dr + \vec{e}_\theta\,r\,d\theta + \vec{e}_\phi\,r\sin(\theta)\,d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

$$\hat{Z}_i(z) = \hat{\eta}_i\,\frac{1 + \hat{\Gamma}_i(z)}{1 - \hat{\Gamma}_i(z)} \quad , \quad \hat{\Gamma}_i(z) = \frac{\hat{Z}_i(z) - \hat{\eta}_i}{\hat{Z}_i(z) - \hat{\eta}_i} \quad \&$$

$\hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\,e^{2\hat{\gamma}_i(z'-z)}$ *where* z' & z *are two positions in* i^{th} *region*