UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2016/2017

TITLE OF PAPER	:	ELECTROMAGNETIC THEORY
COURSE NUMBER	:	P331
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS ELEVEN PAGES, INCLUDING THIS PAGE.

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Question one

(a) A dielectric spherical ball of radius R_0 with a permittivity ε, centered at the origin and embedded in air of permittivity ε_0, carries a volume charge density distribution of $\rho_v = 10\left(1+\alpha\, r^2\right)$ C/m^3 where α is a constant.

(i) Find the total electric charge Q_0 of the dielectric spherical ball in terms of R_0 & α and show that

$$Q_0 = 4\,\pi\left(\frac{10}{3}\,R_0^3 + 2\,\alpha\,R_0^5\right)$$ **(3 marks)**

(Hint : $d\,v = r^2\,\sin(\theta)\,d\,r\,d\,\theta\,d\,\phi$)

(ii) Set $\vec{E} = \vec{e}_r\,E_r(r)$ (make a brief justification of this setting), use the integral form of Gauss's law and choose and draw proper Gaussian surfaces to find $\vec{E}$ in terms of r, R_0 & α for $0 \le r \le R_0$ & $r \ge R_0$ regions. **(1+1+6 marks)**

(iii) Find the value of α in terms of R_0 such that the electric field everywhere outside the spherical ball is zero. **(2 marks)**

(b) A positron of charge $+e$ having an initial constant velocity $\vec{e}_x\,v_0$ and projected into a constant electric and magnetic field $\vec{E} = \vec{e}_x\,E_0$ and $\vec{B} = \vec{e}_y\,B_0$, taking its entrance position as the origin of Cartesian coordinate and entrance moment as $t = 0$, its equation of motion is $m_e\,\dfrac{d\,\vec{v}(t)}{d\,t} = e\,\vec{E} + e\,\vec{v}(t)\times\vec{B}$ ...... (1) where

$\vec{v}(t) = \vec{e}_x\,v_x(t) + \vec{e}_y\,v_y(t) + \vec{e}_z\,v_z(t)$.

(i) Decompose eq.(1) into three scalar differential equations and deduce that

$$\begin{cases} m_e\,\dfrac{d\,v_x(t)}{d\,t} = e\,E_0 - e\,v_z(t)\,B_0 & \cdots\cdots\ (2) \\ m_e\,\dfrac{d\,v_y(t)}{d\,t} = 0 & \cdots\cdots\ (3) \\ m_e\,\dfrac{d\,v_z(t)}{d\,t} = e\,v_x(t)\,B_0 & \cdots\cdots\ (4) \end{cases}$$

(3 marks)

(ii) Eq.(2) & eq.(4) are coupled differential equations for $v_x(t)$ & $v_z(t)$. De-couple them and deduce that

$$\frac{d^2\,v_x(t)}{d\,t^2} = -\,\omega^2\,v_x(t) \quad where \quad \omega = \frac{e\,B_0}{m_e} \quad \cdots\cdots\ (5)$$ **(3 marks)**

Question one (continued)

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(iii) The general solution of $v_x(t)$ from eq.(5) can be written as

$$v_x(t) = k_1 \cos(\omega t) + k_2 \sin(\omega t) \quad \cdots\cdots \quad (6)$$

where k_1 & k_2 are arbitrary constants.

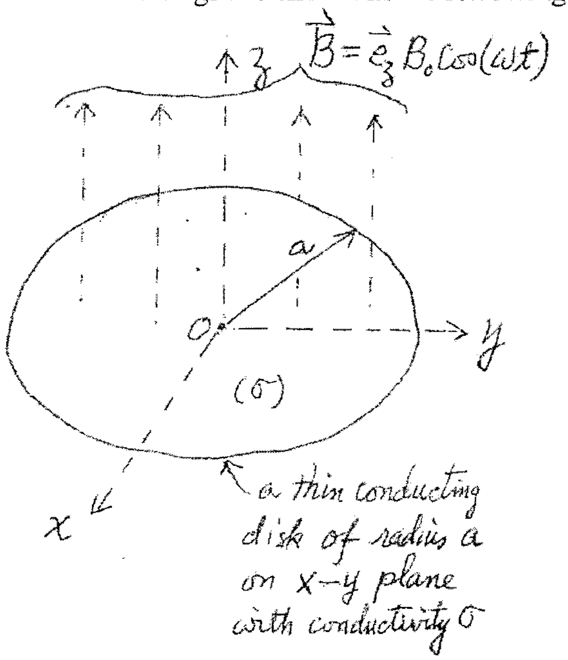
Substitute eq.(6) into eq.(2) and deduce that the general solution of $v_z(t)$ is

$$v_z(t) = \frac{E_0}{B_0} + k_1 \sin(\omega t) - k_2 \cos(\omega t) \quad \cdots\cdots \quad (7) \ .$$ **(2 marks)**

(iv) From the given initial conditions of $v_x(t)$ & $v_z(t)$, i.e., $v_x(0) = v_0$ & $v_z(0) = 0$, and the general solutions of $v_x(t)$ & $v_z(t)$, i.e., eq.(6) & eq.(7) , find the values of k_1 & k_2 in terms of v_0 , E_0 & B_0 and then write down the specific solutions of $v_x(t)$ & $v_z(t)$. **(3+1 marks)**

Question two

(a) A very thin conducting disk of radius a and conductivity σ is placed on *x-y plane* and centred at the origin as shown in the following diagram

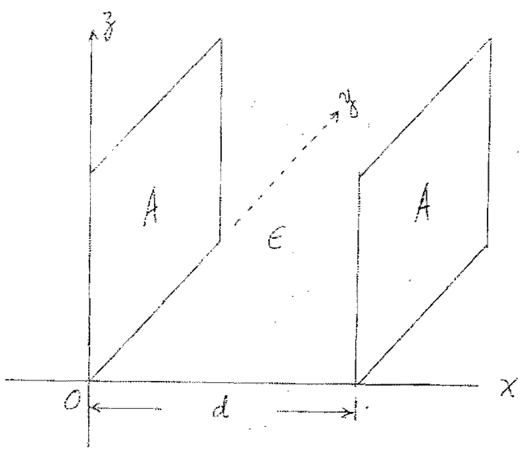


A time-dependent magnetic field $\vec{B}=\vec{a}_z\ B_0 \cos(\omega t)$, where B_0 & ω are constants, is applied to the conducting disk.

(i) Set the induced electric field in the conducting disk as $\vec{E}=\vec{e}_\phi\ E_\phi(\rho)$ (provide a brief justification), use the integral form of Faraday's law, and draw appropriate closed loop to deduce that
$\vec{E}=\vec{e}_\phi\ E_\phi(\rho)$. **(1+1+8 marks)**

(ii) Based on the result of (a)(i), write down the induced current density in the conducting disk. **(1 mark)**

(b) A two-parallel conducting plate capacitor system is shown in the following diagram



where A is the plate surface area, d is the plate separation and ε is the dielectric constant of the insulating material layer in-between the two plates.

Question two (continued)

(i) From $\nabla^2 f(x) = 0$ with boundary conditions of $f(x=0) = 0$ and $f(x=d) = V_0$, find the specific solution of $f(x)$. **(4 marks)**

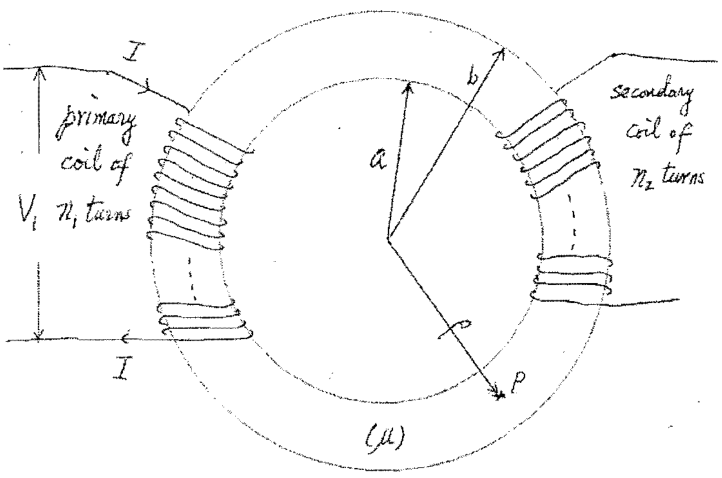
(ii) Find $\vec{E}$ from $f(x)$ obtained in (b)(i) and then write down $\vec{D}$ & $\vec{P}$ in-between the two conducting plates . **(4 marks)**

(iii) Use $\rho_s = D_n$ where ρ_s & D_n are the surface conduction charge density and normal outward $\vec{D}$ component on the conductor's surface respectively , find ρ_s on both $x = 0$ and $x = d$ conductor's surfaces respectively . Then find the total charge on both surfaces with surface area A and show that they are equal and opposite . **(5 marks)**

(iv) Write down the capacitance of this two-parallel conducting plate capacitor system. **(1 mark)**

Question three

A static current I flows in the primary coil of n_1 turn toroid, wired around an iron ring core of magnetic permeability μ with the square cross-section area $(b-a)^2$ as shown below:



where *z-axis* is pointing out of this paper.

(a) Set $\vec{B} = \vec{e}_\phi \; B_\phi(\rho)$ (justify this briefly) and use the integral form of Ampere's law (choose and draw proper closed loops) to find the magnetic field $\vec{B}$ within the iron core , i.e., $a \le \rho \le b$ & $0 \le z \le (b-a)$ region. Show that

$\vec{B} = -\vec{a}_\phi \dfrac{\mu \, n_1 \, I}{2\pi\rho}$ within the iron core. **(1+ 1+5 marks)**

(b) Find the total magnetic flux Ψ_m passing through the cross-section area $(b-a)^2$ of the iron ring in counter clockwise sense, i.e., $\int_S \vec{B} \bullet d\vec{s}$ where $S: a \le \rho \le b$, $0 \le z \le (b-a)$ & $d\vec{s} = \vec{a}_\phi \; d\rho \, dz$, in terms of a, b, n_1, μ & I, i.e.,

show that $\Psi_m = -\dfrac{\mu \, n_1 \, I}{2\pi} \times \ln\left(\dfrac{b}{a}\right) \times (b-a)$ **(4 marks)**

(c) Find the self-inductance L of the primary coil as well as the mutual inductance M of the secondary coil due to the primary coil in terms of a, b, μ, n_1 & n_2.

(5 marks)

(d) (i) If the primary coil carries a sinusoidal current of $I_0 \sin(\omega t)$ instead of carrying a static current I , find the induced e.m.f. $V_2(t)$ in the secondary coil in terms of $a, b, \omega, n_1, n_2, \mu$ & I_0 under quasi static situation . **(5 marks)**

(ii) If the potential drop for the primary coil due to its resistance is negligible compared to the one due to its self-inductance, i.e., $V_1(t) \approx L \dfrac{dI}{dt}$, show that

$\dfrac{|V_2(t)|}{|V_1(t)|} = \dfrac{n_2}{n_1}$. **(4 marks)**

Question four

(a) (i) For any closed surface S, enclosing a volume V, the integral form of the continuity equation for electric charges in Electromagnetic theory can be written as $\oint_S \vec{J} \bullet d\vec{s} = -\frac{d}{dt}\left(\oint_V \rho_v \, dv\right)$. Explain briefly the meaning of the left hand side and the right hand side of this equation and indicate which law in physics it describes. **(3+1 marks)**

(ii) Use the divergence theorem to transform the above integral form of continuity equation for electric charges into its differential form. **(3 marks)**

(iii) (A) Show that without introducing the displacement current term, i.e., $\frac{\partial \vec{D}}{\partial t}$, in the equation for Ampere's law, i.e., $\vec{\nabla} \times \vec{H} = \vec{J}$ instead of $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, Maxwell's equations would contradict the continuity equation for electric charges. **(2 marks)**

(B) Show that by including the displacement current term, Maxwell's equations are in agreement with the continuity equation. **(4 marks)**

(b) (i) From the time-dependent Maxwell's equations deduce the following wave equation for $\vec{E}$ in the material region with parameters of μ, ε & σ where $\rho_v = 0$ & $\vec{J} = \sigma \vec{E}$, as

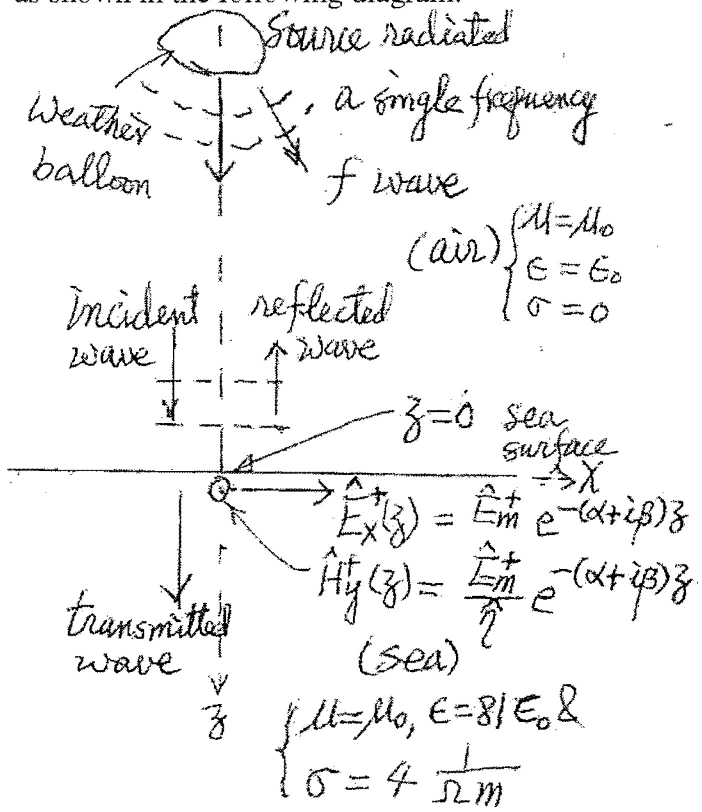
$$\nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$ **(6 marks)**

(ii) By direct substitution, show that $E_x = \hat{E}_m \, e^{\hat{\gamma} z} \, e^{i \omega t}$ (where $\hat{E}_m$ is any constant, ω is any frequency and $\hat{\gamma} = \sqrt{i \omega \mu \sigma - \omega^2 \mu \varepsilon}$) is a solution to the E_x part of the wave equation in (a)(i), i.e., $\nabla^2 E_x = \mu \sigma \frac{\partial E_x}{\partial t} + \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}$.

(6 marks)

Question five

A stationary weather balloon, high above the sea surface (taken as z = 0 plane), radiates a spherical wave of frequency f . When the radiated wave reaches the sea surface directly below the balloon , it can be considered as a uniform plane wave incident normally upon the sea surface, locally at least, as shown in the following diagram.



Assuming the sea water has the constants $\mu = \mu_0$, $\varepsilon = 81\,\varepsilon_0$ & $\sigma = 4 \quad \Omega^{-1}\mathrm{m}^{-1}$ and $\hat{E}_x^+(0) = \hat{E}_m^+ = 1 \quad$ V/m.

(a) If $f = 10^4$ Hz (i.e., in the very low frequency VLF range) ,

(i) find the values of the loss tangent $\dfrac{\sigma}{\omega\,\varepsilon}$, the propagation constant $\hat{\gamma}\ (= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave at this frequency in the sea, **(2+2+2 marks)**

(ii) express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (a)(i) inserted for this transmitted wave, and **(6 marks)**

(iii) find the values of the penetration depth , the wave length and phase velocity of the given wave at this frequency in the sea. **(3 marks)**

(b) If $f = 1000$ Hz (i.e., in the extremely low frequency ELF range),

(i) find the values of the loss tangent $\frac{\sigma}{\omega \varepsilon}$, the propagation constant $\hat{\gamma}$ $(= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave at this frequency in the sea, and **(2+2+2 marks)**

(ii) find the values of the penetration depth, the wave length and phase velocity of the given wave at this frequency in the sea. **(3 marks)**

(c) Based on the results of (a)(iii) and (b)(ii), comment on the effectiveness of undersea radio communication. **(1 mark)**

Useful informations

$$e = 1.6 \times 10^{-19}\ C$$

$$m_e = 9.1 \times 10^{-31}\ kg$$

$$\mu_0 = 4\,\pi \times 10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\,\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\ \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega\,\varepsilon}\right)^2} - 1}$$

$$\beta = \frac{\omega\,\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\ \sqrt{\sqrt{1 + \left(\frac{\sigma}{\omega\,\varepsilon}\right)^2} + 1}$$

$$\frac{1}{\sqrt{\mu_0\ \varepsilon_0}} = 3 \times 10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\ e^{\,i\,\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\,\pi\quad \Omega = 377\quad \Omega$$

$$\beta_0 = \omega\,\sqrt{\mu_0\ \varepsilon_0}$$

$$\oiint_S \vec{E} \bullet d\vec{s} = \frac{1}{\varepsilon} \iiint_V \rho_v\ dv$$

$$\oiint_S \vec{B} \bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B} \bullet d\vec{s}\right)$$

$$\oint_L \vec{B} \bullet d\vec{l} = \mu \iint_S \vec{J} \bullet d\vec{s} + \mu\,\varepsilon\,\frac{\partial}{\partial t}\left(\iint_S \vec{E} \bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\,\frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

$$\vec{D} = \varepsilon\, \vec{E} = \varepsilon_0\, \vec{E} + \vec{P} \qquad \& \qquad \vec{B} = \mu\, \vec{H} = \mu_0\, \vec{H} + \vec{M}$$

$$\oint\!\!\!\oint_S \vec{F} \bullet d\vec{s} \equiv \oint\!\!\!\oint\!\!\!\oint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla}\, f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2\, \vec{F}$$

$$\vec{\nabla}\, f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial (F_x)}{\partial x} + \frac{\partial (F_y)}{\partial y} + \frac{\partial (F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial (F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial (F_\phi)}{\partial \phi} + \frac{\partial (F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial (F_r\, r^2)}{\partial r} + \frac{1}{r \sin(\theta)}\frac{\partial (F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r \sin(\theta)}\frac{\partial (F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial (F_z)}{\partial y} - \frac{\partial (F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial (F_x)}{\partial z} - \frac{\partial (F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial (F_y)}{\partial x} - \frac{\partial (F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial (F_z)}{\partial \phi} - \frac{\partial (F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial (F_\rho)}{\partial z} - \frac{\partial (F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial (F_\phi\, \rho)}{\partial \rho} - \frac{\partial (F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)}\left(\frac{\partial (F_\phi\, r \sin(\theta))}{\partial \theta} - \frac{\partial (F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r \sin(\theta)}\left(\frac{\partial (F_r)}{\partial \phi} - \frac{\partial (F_\phi\, r \sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial (F_\theta\, r)}{\partial r} - \frac{\partial (F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x\, F_x + \vec{e}_y\, F_y + \vec{e}_z\, F_z = \vec{e}_\rho\, F_\rho + \vec{e}_\phi\, F_\phi + \vec{e}_z\, F_z = \vec{e}_r\, F_r + \vec{e}_\theta\, F_\theta + \vec{e}_\phi\, F_\phi$ and

$$d\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r \sin(\theta)\, d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

$$\hat{Z}_i(z) = \hat{\eta}_i \frac{1 + \hat{\Gamma}_i(z)}{1 - \hat{\Gamma}_i(z)} \quad , \quad \hat{\Gamma}_i(z) = \frac{\hat{Z}_i(z) - \hat{\eta}_i}{\hat{Z}_i(z) - \hat{\eta}_i} \quad \&$$

$\hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2 \hat{\gamma}_i (z' - z)}$ *where* z' & z *are two positions in* i^{th} *region*