UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

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DEPARTMENT OF PHYSICS

MAIN EXAMINATION: 2016/2017

TITLE OF THE PAPER: QUANTUM MECHANICS

COURSE NUMBER: P342

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

- ANSWER ANY FOUR OUT THE FIVE QUESTIONS.
- EACH QUESTION CARRIES 25 MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN PAGE 2 WHEN NECESSARY.

THIS PAPER HAS 7 PAGES, INCLUDING THIS PAGE.

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Useful Formulas

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Time-dependent Schrodinger equation: $i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t)$

Time-independent Schrodinger equation: $\hat{H}\psi(x) = E\psi(x)$

Hamiltonian operator : $\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + V(x)$

Momentum operator $\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\psi(x)$

Probability current $J(x,t) = \frac{i\hbar}{2m}\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right)$

Fourier transform $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ikx}dk$

Inverse Fourier transform $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx$

$\int_{-\infty}^{\infty} x^{2n}e^{-ax^2}dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}$, n =0, 1, 2, 3, ...

$\int_{-\infty}^{\infty} x^{2n+1}e^{-ax^2}dx = 0$, n =0, 1, 2, 3, ...

Heisenberg uncertainty $\Delta x\Delta p \geq \hbar/2$

Uncertainty of a quantity $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$

Raising and Lowering operators: $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

Infinite potential well:

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{elsewhere} \end{cases}$$

$E_n = \frac{\hbar^2\pi^2n^2}{2ma^2}$ and $u_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$

Trigonometry:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Question 1

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At time $t = 0$ a particle of mass m trapped in an infinite square well of width L is in a superposition of ground state and two higher energy states,

$$\psi_s(x,0) = A[u_1(x) + \sqrt{2}u_2(x) + \sqrt{3}u_3(x)]$$

where the $u_n(x)$ are correctly-normalized energy eigenstates with energies E_n.

(a) Which of the following values of A give a properly normalized wavefunction?

i) $\frac{1}{\sqrt{6}}$ ii) $\frac{i}{6}$ iii) $\frac{-i}{\sqrt{3}}$ iv) $\frac{1}{3}$ v) None of these

[5 marks]

(b) Given the wavefunction ψ_s, what is the probability that the energy is E_3 at $t = 0$? Explain.

i) 0 ii) 1/6 iii) 1/3 iv) 1/2 v) 1

[5 marks]

Let $\phi_{nlm}(\mathbf{r})$ denote the ortho-normalized energy eigenfunctions of the Coulomb potential with principal quantum number n and angular momentum quantum numbers l and m. Consider the state $\psi(\mathbf{r}) = C[\phi_{100}(\mathbf{r}) + 4i\phi_{210}(\mathbf{r}) - 2\sqrt{2}\phi_{21-1}(\mathbf{r})]$.

(c) Which of the following values of C give a properly normalized wavefunction $\phi_{nlm}(\mathbf{r})$?

i) $\frac{1}{\sqrt{5}}$ ii) $\frac{i}{5}$ iii) $\frac{1}{5}$ iv) $\frac{-i}{5}$ v) None of these

[5 marks]

(d) The expectation value of the z-component of the particle angular momentum $\hat{L}_z$ is

i) 0 ii) $80\hbar^2/25$ iii) 1/5 iv) $-8\hbar/25$ v) None of these

[5 marks]

(e) Let ϕ_n be the properly-normalized n^{th} energy eigenfunction of the harmonic oscillator, and let $\psi = (\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger)\phi_n$. Which of the following is equal to ψ?

i) 0 ii) $2n\phi_{n-1}$ iii) $(n+1)\phi_n$ iv) $(2n+1)\phi_n$ v) None of these

[5 marks]

NB: For full marks please provide a detailed working that supports your answer.

Question 2

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Consider a particle of mass m inside a box of size a with infinite walls

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty, & \text{elsewhere} \end{cases}$$

The wavefunction is specified at $t = 0$ to be $\psi(x, 0) = N[\sin(2kx) - 2\sin(3kx)]$, where $k = \pi/a$.

(a) Expand the wavefunction at the initial time $\psi(x, t = 0)$ in term of the orthonormal eigenfunctions $u_n(x)$ as given in the formula section.

[5 marks]

(b) Determine the normalization constant N.

[5 marks]

(c) Write down $\psi(x, t)$ at an arbitrary later time t.

[5 marks]

(d) If a measurement of the particle's energy at time t is performed, what will be the possible outcomes, and with what probability will those values be measured?

[5 marks]

(e) What is the average energy $\langle E \rangle$ of the particle in the box? Does $\langle E \rangle$ change over time?

[5 marks]

Question 3

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(a) A particle of mass m in a one dimensional simple harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$ has the initial wavefunction

$$|\psi, t = 0\rangle = N(|1\rangle - \sqrt{5}i|2\rangle)$$

where $|n\rangle$, $n = 0, 1, ...$, are the harmonic oscillator energy eigenstates, satisfying $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ where $\hat{a}^\dagger$ and $\hat{a}$ are the raising and lowering operators respectively. In term of $\hat{a}^\dagger$ and $\hat{a}$, the Hamiltonian operator of this system $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

(i) Find N such that $|\psi, t = 0\rangle$ is properly normalized.

[5 marks]

(ii) What is the wavefunction at a later time, $|\psi, t\rangle$?

[3 marks]

(iii) Compute the expectation value of the energy $\langle\psi, t|\hat{H}|\psi, t\rangle$?

[5 marks]

(iv) If a measurement of the electron's energy were carried out what values could be found and with what probabilities?

[5 marks]

(b) A simple 1D harmonic oscillator is a particle acted upon by a linear restoring force $F = -m\omega^2x$. Classically, the minimum energy of the oscillator is zero because we can place it precisely at $x = 0$, its equilibrium position, while giving it zero initial velocity. Quantum mechanically, the uncertainty principle does not allow us to localize the particle precisely and simultaneously have it at rest. Using the uncertainty principle to estimate the minimum energy of the quantum mechanical oscillator.

[7 marks]

Question 4

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(a) Using $[\hat{x}, \hat{p}] = i\hbar$, show that

$$[\hat{x}^2, \hat{p}] = 2i\hbar\hat{x}, \quad [\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}, \quad \text{and} \quad [\hat{x}^2, \hat{p}] = 2i\hbar(i\hbar + 2\hat{p}\hat{x})$$

[7 marks]

(b) The angular momentum operator $\hat{L} = (L_x, L_y, L_z)$, where $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, and $L_z = xp_y - yp_x$. Which of the following quantities describing a particle's motion in a central potential in 3-dimensions *cannot* be simultaneously measured with arbitrary accuracy, even in principle:

(i) y position and the z-component of angular momentum.

[5 marks]

(ii) x position and the x-component of the angular momentum.

[5 marks]

(c) An electron in the Coulomb field of a proton is in a state described by the wavefunction

$$\psi(\vec{r}) = \frac{1}{\sqrt{10}}(2\phi_{100} + \phi_{210} + \sqrt{2}\phi_{211} + \sqrt{3}\phi_{21-1})$$

where the subscripts are the values of the quantum number n, l, and m, respectively. Note that $L_z\phi_{nlm} = m\hbar\phi_{nlm}$. If the z-component of the electron's angular momentum L_z were measured, what values would one obtain, and with what probabilities?

[8 marks]

Question 5

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Consider a neutron which is confined to an infinite potential well of width $a = 8$fm. At time $t = 0$ the neutron is assumed to be in the state

$$\psi(x,0) = \sqrt{\frac{4}{7a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{7a}} \sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{8}{7a}} \sin\left(\frac{3\pi x}{a}\right)$$

(a) If an energy measurement is carried out on the system, what are the values that will be found for the energy and with what probabilities? Express your answer in MeV. NB: the rest mass of the neutron $mc^2 = 939$MeV and $\hbar c = 197$ MeVfm.

[15 marks]

(b) If the measurement is repeated on many identical systems, what is the average value of the energy that will be found? Again, express your answer in MeV

[5 marks]

(c) Using the uncertainty principle, estimate the order of magnitude of the neutron speed in this well as a function of the speed of light.

[5 marks]