UNIVERSITY OF SWAZILAND 27

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION: 2016/2017

TITLE OF PAPER: ELECTRICITY AND MAGNETISM

COURSE NUMBER: PHY221/P221

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

- ANSWER ANY FOUR OUT OF THE FIVE QUESTIONS.
- EACH QUESTION CARRIES 25 POINTS.
- POINTS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION IN THE NEXT PAGE WHEN NECESSARY.

THIS PAPER HAS 7 PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THIS PAGE UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

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Useful Mathematical Relations

Gradient Theorem

$$\int_{\vec{a}}^{\vec{b}} (\nabla f) \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$$

Divergence Theorem

$$\int \nabla \cdot \vec{A} d\tau = \oint \vec{A} \cdot d\vec{a}$$

Curl Theorem

$$\int (\nabla \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$$

Line and Volume Elements

Cartesian: $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, $d\tau = dxdydz$

Cylindrical: $d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}$, $d\tau = sdsd\phi dz$

Spherical: $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$, $d\tau = r^2 \sin\theta dr d\theta d\phi$

Gradient and Divergence in Spherical Coordinates

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Dirac Delta Function

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

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Question 1: ELECTROSTATICS..

Consider an infinite line charge with constant linear charge density λ. Choose the z-axis to concide with the charge distribution.

(a) For a given field point show that the component of the electrostatic field parallel to the distribution is zero. (5)

(b) For a given field point, show that the electrostatic field is given by (8)

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s},$$

where s is the distance from the charge distribution.

(c) Use the electrostatic field and the definition $V(\vec{r}) = -\int_O^{\vec{r}} \mathbf{E} \cdot d\mathbf{l}$, where O is the reference, to determine the electrostatic potential $V(\vec{r})$. (8)

(d) What is the anomaly with the electrostatic potential at infinity? Why is there an anomaly? (3)

(e) What is the appropriate reference point for the potential? (1)

You may need the integral $\int \frac{dx}{(s^2+x^2)^{3/2}} = \frac{x}{s^2(s^2+x^2)^{1/2}} +$ constant

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Question 2: ELECTROSTATIC II ....

Figure 1 shows a spherical capacitor in which a positive charge $+Q$ has been deposited on the inner sphere of radius a and a negative charge $-Q$ has been deposited on the outer spherical shell of inner radius b. The region between the sphere and spherical shell is empty.

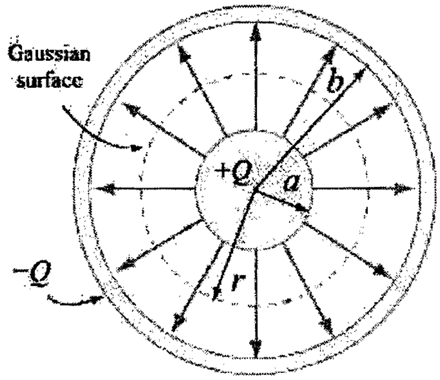

Figure 1: Cross-sectional view of a spherical capacitor.

(a) What are the equipotential surfaces in this configuration? (2)

(b) Use Gauss law to determine the electrostatic field at any point in the region between the sphere and the spherical shell, i.e a position $\vec{r}$ such that $a < r < b$. (7)

(c) Use the field to determine the potential difference between the two surfaces. (8)

(d) Use the potential difference to calculate the capacitance. (4)

(e) Hence, determine the electrostatic energy stored in the capacitor. (4)

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Question 3: Magnetostatics..

Consider a finite segment of wire lying along the z-axis carrying a uniform current I flowing in the positive z-direction. Assume the wire to be of length $2L$, with the center at $z = 0$.

(a) For a given field point P a distance s from the wire, $z = 0$ and arbitrary value of the ϕ coordinate, draw a diagram depicting the field point and an infinitesimal current/line element you can use to determine the magnetostatic potential. (1)

(b) Using the diagram and the Biot-Savart law, determine the magnetostatic potential at the field point P. (8)

(c) Use the magnetostatic potential to determine the magnetostatic field consistent with it. (Show full details.) (8)

(d) Take the infinite limit of your result for the field and then use Ampere's law to verify it. (8)

Note

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \ln\left[2\left(x + \sqrt{a^2 + x^2}\right)\right]$$

In cylindrical coordinates:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z}$$

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Question 4: Magnetostatics II ....

(a) For charge flowing through a volume, the current density **J** is defined as the current per unit area perpendicular to flow. Local charge conservation results in the continuity equation (8)

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

where ρ is the volume charge density. Use charge conservation and the appropriate definitions to derive the continuity equation.

(b) At a given instant, a certain system has a current density given by $\mathbf{J} = k(x^3\hat{x} + y^3\hat{y} + z^3\hat{z})$, where k is a positive constant.

i. In what units will k be measured in? (2)

ii. At this instant, what is the rate of change of the charge density at the point $(2, -1, 4)$ meter? (5)

iii. Consider the total charge Q contained within a sphere of radius a centered at the origin. At this instant, what is the rate at which Q is changing in time? Is Q decreasing or increasing? (10)

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Question 5: Induction and Alternating Current Circuits ....

A transformer, such as the one in figure 2 takes an AC input voltage of amplitude V_1, and delivers an output voltage V_2, which is determined by the turns ratio $\frac{V_2}{V_1} = \frac{N_2}{N_1}$. The transformer works as voltage step up or a voltage step down. This question is intended to show that energy conservation is not violated in either scenario.

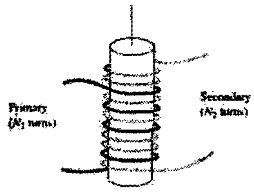

Figure 2: A simple transformer

(a) In an ideal transformer the same flux passes through all turns of the primary and secondary. Show that the relationship between the mutual inductance M and the self inductances L_1 and L_2 is $M^2 = L_1 L_2$. (3)

(b) Suppose the primary voltage is given by $V_{in} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, R. Show that the currents I_1 and I_2 satisfy the relations. (4)

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t)$$
$$L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R$$

(c) Assuming no DC component in I_1, solve the two equations for $I_1(t)$ and $I_2(t)$. (8)

(d) Calculte the input power ($P_{in} = V_{in} I_1$) and the output power ($P_{out} = V_{out} I_2$) and show that their averages over a full cycle are equal. (10)