UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

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MAIN EXAMINATION 2016/2017

TITLE OF PAPER : MATHEMATICAL METHODS FOR PHYSICISTS

COURSE NUMBER : P272/PHY271

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.
MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS SEVEN PAGES, INCLUDING THIS PAGE.

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Question one

(a) Given an arbitrary scalar and continuous function in cylindrical coordinates as $f(\rho,\phi,z)$, prove that $\vec{\nabla}\times\left(\vec{\nabla} f(\rho,\phi,z)\right)\equiv 0$. **(7 marks)**

(b) Given $\vec{F}=\vec{e}_x\left(y^2\right)+\vec{e}_y\left(2\,x\,z\right)+\vec{e}_z\left(3\,x^2\right)$ in Cartesian coordinates, find the value of the following line integral

$\int_{P_1,L}^{P_2}\vec{F}\bullet d\vec{l}$ if $P_1:(3,0,2)$, $P_2:(6,27,2)$ and

(i) L : a straight line from P_1 to P_2 on $z=2$ plane . **(7 marks)**

(ii) L : a parabolic path described by $y=x^2-9$ from P_1 to P_2 on $z=2$ plane . Compare this answer with that obtained in (b)(i) and comment on whether the given $\vec{F}$ is a conservative vector field or not. **(7 marks)**

(iii) Find $\vec{\nabla}\times\vec{F}$. Does this result agree with the comment you made in (b)(ii)? **(4 marks)**

Question two

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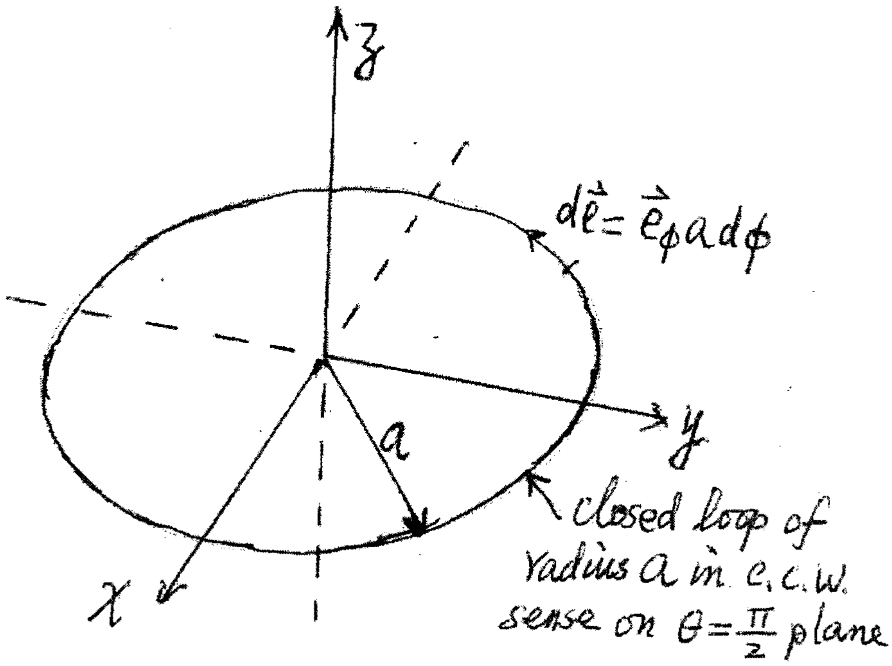
Given $\vec{F} = \vec{e}_r \left(r^3 \cos(\theta)\right) + \vec{e}_\theta \left(2r^3 \sin(\phi)\right) + \vec{e}_\phi \left(3\, r^3 \sin(\theta)\right)$ in spherical coordinates,

(a) find the value of $\oint_L \vec{F} \bullet d\vec{l}$ if L is the circular closed loop of radius a on $\theta = \frac{\pi}{2}$ *plane* in counter clockwise sense as shown in the diagram below



i.e.,

$$L : \left(r = a \;,\; \theta = \frac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{l} = +\vec{e}_\phi\, r \sin(\theta)\, d\phi \xrightarrow{r = a \;\&\; \theta = \frac{\pi}{2}} \vec{e}_\phi\, a\, d\phi \right)$$

(7 marks)

(b) (i) find $\vec{\nabla} \times \vec{F}$, **(7 marks)**

(ii) then evaluate the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by L given in (a) , i.e.,

$$S : \left(0 \le r \le a \;,\; \theta = \frac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = -\vec{e}_\theta\, r \sin(\theta)\, dr\, d\phi \xrightarrow{\theta = \frac{\pi}{2}} -\vec{e}_\theta\, r\, dr\, d\phi \right)$$

Compare this value with that obtained in (a) and make a brief comment.

(11 marks)

Question three

u6

Given the following non-homogeneous differential equation as $\frac{d^2 x(t)}{dt^2} + 4\,x(t) = f(t)$, where $f(t)$ is a periodic rectangular barrier shape driving force of period 10 , i.e., $f(t) = f(t+10) = f(t+20) = \cdots\cdots$., and its first period description is

$$f(t) = \begin{cases} k & for \quad 0 \le t \le 5 \\ 0 & for \quad 5 \le t \le 10 \end{cases} \qquad \text{where } k \text{ is a constant,}$$

(a) find the Fourier series expansion of $f(t)$ and show that

$$f(t) = \frac{k}{2} + \sum_{n=1}^{\infty} \frac{k\,(1-\cos(n\pi))}{n\,\pi} \sin\left(\frac{n\,\pi\,t}{5}\right) \quad \cdots\cdots \quad (1)$$

(10 marks)

(b) find the particular solution of the given non-homogeneous differential equation $x_p(t)$ and show that

$$x_p(t) = \frac{k}{8} + \sum_{n=1}^{\infty} \left\{ \frac{k\,(1-\cos(n\pi))}{n\,\pi\left(-\frac{n^2\,\pi^2}{25}+4\right)} \sin\left(\frac{n\,\pi\,t}{5}\right) \right\} \quad \cdots\cdots \quad (2)$$

(12 marks)

(c) for the homogeneous part of the given non-homogeneous differential equation , i.e., $\frac{d^2 x(t)}{dt^2} + 4\,x(t) = 0$, set $x(t) = e^{\alpha t}$ and find the appropriate values of α and thus write down its general solution $x_h(t)$. Then write down the general solution of the given non-homogeneous differential equation $x_g(t)$. **(2+1 marks)**

Question four 47

(a) Given the following 2-D Laplace equation in cylindrical coordinates as

$$\nabla^2 f(\rho,\phi) = 0 = \frac{\partial^2 f(\rho,\phi)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f(\rho,\phi)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f(\rho,\phi)}{\partial \phi^2} ,$$

(i) set $f(\rho,\phi) = F(\rho)\,G(\phi)$ and use separation variable scheme to separate the above partial differential equation into the following two ordinary differential equations.

$$\begin{cases} \rho^2 \dfrac{d^2(F(\rho))}{d\rho^2} + \rho \dfrac{d(F(\rho))}{d\rho} = k\,F(\rho) & \cdots\cdots \quad (1) \\ \dfrac{d^2(G(\phi))}{d\phi^2} = -\,k\,G(\phi) & \cdots\cdots \quad (2) \end{cases}$$

where k is a separation constant **(5 marks)**

(ii) From eq.(2), explain briefly why the appropriate values for k are m^2 where $m = 0,1,2,3,\cdots\cdots$. **(2 marks)**

(b) If $m = 2$ thus eq.(2) in (a)(i) becomes $\dfrac{d^2(G(\phi))}{d\phi^2} + 4\,G(\phi) = 0$.

Set $G(\phi) = \sum_{n=0}^{\infty} a_n\,\phi^{n+s}$ & $a_0 \neq 0$ and utilize the power series method,

(i) write down its indicial equations and show that $s = 0$ *or* 1 and $a_1 = 0$, **(7 marks)**

(ii) For $s = 0$ independent solution, named as $G_1(\phi)$, write down its recurrence relation. Set $a_0 = 1$ and use the recurrence relation to generate $G_1(\phi)$ in power series form truncated up to a_6 term. **(7 marks)**

(iii) Show that $G_1(\phi)$ is linearly dependent on the two closed form independent solutions of eq.(2) which are $\sin(2\phi)$ & $\cos(2\phi)$. **(4 marks)**

(Hint : $$\begin{cases} \sin(x) = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + O(x^9) \\ \cos(x) = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + O(x^8) \end{cases}$$)

Question five

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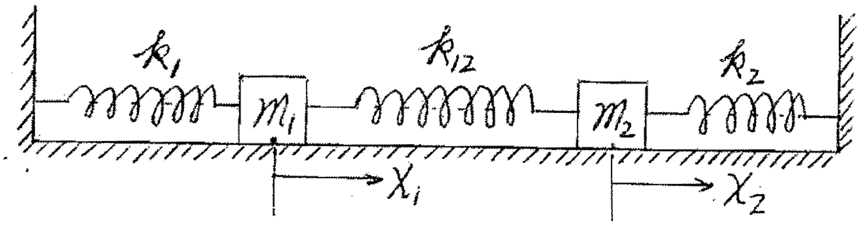
Two simple harmonic oscillators are joined by a spring with a spring constant k_{12} as shown in the diagram below :



The equations of motion for this coupled oscillator system ignoring friction are given as

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -\left(k_1 + k_{12}\right) x_1(t) + k_{12}\, x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = k_{12}\, x_1(t) - \left(k_2 + k_{12}\right) x_2(t) \end{cases}$$

where x_1 & x_2 are horizontal displacements of m_1 & m_2 measured from their respective resting positions.

If given $m_1 = 1\ kg$, $m_2 = 3\ kg$, $k_1 = 2\ \dfrac{N}{m}$, $k_2 = 6\ \dfrac{N}{m}$ & $k_{12} = 3\ \dfrac{N}{m}$,

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, then the above given equations can be deduced to the following matrix equation $A\, X = -\omega^2\, X$ where

$$A = \begin{pmatrix} -5 & 3 \\ 1 & -3 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(5 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(6 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b), **(6 marks)**

(d) find the normal coordinates of the given coupled system , **(6 marks)**

(e) write down the general solutions for $x_1(t)$ & $x_2(t)$. **(2 marks)**

Useful informations

u9

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$f(t) = f(t+2L) = f(t+4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \textit{where}$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \quad \textit{for } n = 1,2,3,\cdots$$

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$