UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2016/2017

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272/PHY271
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

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Question one

Given $\vec{F} = \vec{e}_\rho \left(4\rho^3\right) + \vec{e}_\phi \left(2\rho^3 \cos\phi\right) + \vec{e}_z \left(\rho z^2\right)$ in cylindrical coordinates,

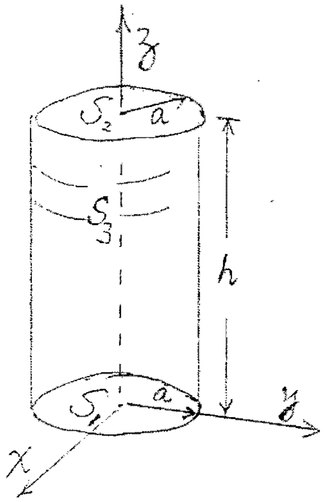
(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if S is the closed surface enclosing the cylindrical tube of cross-sectional radius a and tube height h, i.e., $S = S_1 + S_2 + S_3$ where

S_1 : $\left(z = 0 \ , \ 0 \le \rho \le a \ , \ 0 \le \phi \le 2\pi \ \ \& \ \ d\vec{s} = -\vec{e}_z \ \rho \, d\rho \, d\phi \right)$

S_2 : $\left(z = h \ , \ 0 \le \rho \le a \ , \ 0 \le \phi \le 2\pi \ \ \& \ \ d\vec{s} = +\vec{e}_z \ \rho \, d\rho \, d\phi \right)$

S_3 : $\left(\rho = a \ , \ 0 \le \phi \le 2\pi \ , \ 0 \le z \le h \ \ \& \ \ d\vec{s} = \vec{e}_\rho \ \rho \, d\phi \, dz \xrightarrow{\rho = a} \vec{e}_\rho \ a \, d\phi \, dz \right)$

The chosen closed surface S is shown in the diagram below :



(12 marks)

(b) (i) find $\vec{\nabla} \bullet \vec{F}$, **(4 marks)**

(ii) then evaluate the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in (a) , i.e., V : $0 \le \rho \le a \ , \ 0 \le \phi \le 2\pi \ , \ 0 \le z \le h \ \ \& \ \ dv = \rho \, d\rho \, d\phi \, dz$.

Compare this value with that obtained in (a) and make a brief comment.

(9 marks)

Question two

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Given the following non-homogeneous differential equation as

$$\frac{d^2\,x(t)}{d\,t^2} - \frac{d\,x(t)}{d\,t} - 2\,x(t) = 20\,e^{-3t} + 5\cos(2t)\ ,$$

(a) set its particular solution as $x_p(t) = k_1\,e^{-3t} + k_2\,\sin(2t) + k_3\,\cos(2t)$ and find the values of k_1 , k_2 & k_3 . **(10 marks)**

(b) for the homogeneous part of the given non-homogeneous differential equation , i.e.,

$\frac{d^2\,x(t)}{d\,t^2} - \frac{d\,x(t)}{d\,t} + 2\,x(t) = 0$, set $x(t) = e^{\alpha t}$ and find the appropriate values of α and thus write down its general solution $x_h(t)$ **(5 marks)**

(c) write down the general solution of the given non-homogeneous differential equation in terms of the answers obtained in (a) & (b) . If the initial conditions are

$x(0) = 6$ & $\left.\frac{d\,x(t)}{d\,t}\right|_{t=0} = -1$, find its specific solution $x_s(t)$. **(10 marks)**

Question three

Given the following Legendre's differential equation as :

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$$\left(1-x^2\right)\frac{d^2 y(x)}{dx^2} - 2x\frac{d\,y(x)}{dx} + 12\,y(x) = 0 \quad \cdots\cdots \quad (1)$$

use the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$;

(a) write down the indicial equations. Find the values of s and a_1 . Show that a_1 can be zero resulting from the indicial equations and thus use $a_1 = 0$ for the subsequent calculations in (b). **(10 marks)**

(b) write down the recurrence relation. For all the appropriate values of s found in (a), set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_6 . Thus write down two independent solution in their power series forms and show that one of the solutions is a polynomial. **(15 marks)**

Question four

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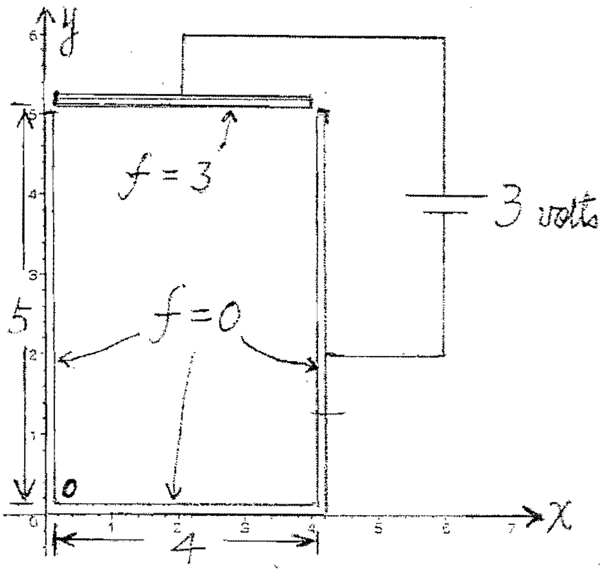
An U – tube capacitor extended very long into z direction with its x – y cross section of area 4×5 as shown below :



Its electric potential $f(x,y)$ for the space in-between the two conductors, i.e., $0 < x < 4$ & $0 < y < 5$, satisfies the following two dimensional Laplace equation :

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0 \quad \cdots\cdots \quad (1)$$

If connecting *3 volts* battery to the given capacitor, the boundary conditions for the capacitor are

$f(0,y) = 0$, $f(4,y) = 0$, $f(x,0) = 0$ & $f(x,5) = 3$

(a) set $f(x,y) = F(x)\,G(y)$ and use separation scheme to deduce the following ordinary differential equations :

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = -k^2 F(x) \\ \dfrac{d^2 G(y)}{d y^2} = +k^2 G(y) \end{cases}$$

where k is a separation constant. **(5 marks)**

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(b) write the general solution for (a) as

$$f(x,y)=\sum_{\forall k} f_k(x,y)$$
$$=\sum_{\forall k}\left(A_k\ \cos(k\,x)+B_k\ \sin(k\,x)\right)\left(C_k\ \cosh(k\,y)+D_k\ \sinh(k\,y)\right)$$

where A_k , B_k , C_k & D_k are arbitrary constants.

Apply three zero boundary conditions which are equivalent to

$f_k(0,y)=0$, $f_k(4,y)=0$ & $f_k(x,0)=0$, and show that the general solution can be simplified as :

$$f(x,y)=\sum_{n=1}^{\infty} E_n\ \sin\left(\frac{n\pi x}{4}\right)\sinh\left(\frac{n\pi y}{4}\right)$$

where E_n $n=1,2,3,\cdots\cdots$ are arbitrary constants. **(10 marks)**

(c) Apply the non-zero boundary condition, i.e., $f(x,5)=3$, deduce that

$$E_n=\frac{6\left(1-\cos(n\pi)\right)}{n\pi\ \sinh\left(\frac{5n\pi}{4}\right)} \qquad n=1,2,3,\cdots\cdots$$ **(10 marks)**

(Hint : $\int_0^4 \sin\left(\frac{n\pi x}{4}\right)\sin\left(\frac{m\pi x}{4}\right)dx=2\,\delta_{n,m}=\begin{cases}0 & if \quad n\neq m\\ 2 & if \quad n=m\end{cases}$)

Question five

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Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -6\, x_1(t) + 3\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 4\, x_1(t) - 5\, x_2(t) \end{cases}$$

(a) Set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation

$$A\,X = -\,\omega^2\, X \quad \text{where} \quad A = \begin{pmatrix} -6 & 3 \\ 4 & -5 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(4 marks)**

(b) Find the eigenfrequencies ω_1 & ω_2 of the given coupled system . **(6 marks)**

(c) Find the eigenvectors $\overline{X}_1$ & $\overline{X}_2$ of the given coupled system corresponding to each eigenfrequencies ω_1 & ω_2 found in (b) respectively. **(6 marks)**

(d) Set $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, then the general solution of $x(t)$ in terms of the eigenfrequencies and eigenvectors found in (b) and (c) can be written as

$$x(t) = k_1\, \cos(\omega_1\, t)\, \overline{X}_1 + k_2\, \sin(\omega_1\, t)\, \overline{X}_1 + k_3\, \cos(\omega_2\, t)\, \overline{X}_2 + k_4\, \sin(\omega_2\, t)\, \overline{X}_2$$ where

k_1 , k_2 , k_3 & k_4 are arbitrary constants.

If the initial conditions of the system are given as :

$x_1(0) = 2$, $x_2(0) = 0$, $\dot{x}_1(0) = -1$ & $\dot{x}_2(0) = 0$, then find the specific values of k_1 , k_2 , k_3 & k_4 which satisfies the given initial conditions. **(9 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\,\sin(\theta)\,\cos(\phi) \\ y = r\,\sin(\theta)\,\sin(\phi) \\ z = r\,\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\,\cos(\phi) \\ y = \rho\,\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1\,h_2\,h_3}\left(\frac{\partial(F_1\,h_2\,h_3)}{\partial u_1} + \frac{\partial(F_2\,h_1\,h_3)}{\partial u_2} + \frac{\partial(F_3\,h_1\,h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2\,h_3}\left(\frac{\partial(F_3\,h_3)}{\partial u_2} - \frac{\partial(F_2\,h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1\,h_3}\left(\frac{\partial(F_1\,h_1)}{\partial u_3} - \frac{\partial(F_3\,h_3)}{\partial u_1}\right)$$
$$+ \frac{\vec{e}_3}{h_1\,h_2}\left(\frac{\partial(F_2\,h_2)}{\partial u_1} - \frac{\partial(F_1\,h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1\,F_1 + \vec{e}_2\,F_2 + \vec{e}_3\,F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\,\sin(\theta))$	for spherical coordinate system