UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2017/2018

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272/PHY271
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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Question one

(a) Given the following relations between the unit vectors of cylindrical, spherical and Cartesian coordinate systems as

$$\begin{cases} \vec{e}_\rho = \vec{e}_x \cos(\phi) + \vec{e}_y \sin(\phi) \\ \vec{e}_\phi = -\vec{e}_x \sin(\phi) + \vec{e}_y \cos(\phi) \end{cases} \quad \& \quad \begin{cases} \vec{e}_r = \vec{e}_\rho \sin(\theta) + \vec{e}_z \cos(\theta) \\ \vec{e}_\theta = \vec{e}_\rho \cos(\theta) - \vec{e}_z \sin(\theta) \end{cases},$$

and deduce the following:

(i) $\dfrac{d\vec{e}_\phi}{dt} = -\vec{e}_\rho \dfrac{d\phi}{dt}$ in terms of cylindrical unit vectors ; **(3 marks)**

(ii) $\dfrac{d\vec{e}_\phi}{dt} = -\vec{e}_r \sin(\theta)\dfrac{d\phi}{dt} - \vec{e}_\theta \cos(\theta)\dfrac{d\phi}{dt}$ in terms of spherical unit vectors . **(4 marks)**

(b) Given $\vec{F} = \vec{e}_x\,(3x^2 + yz) + \vec{e}_y\,(xz) + \vec{e}_z\,(xy)$ and find the value of $\int_{P_1,L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (0, 0, 2)$, $P_2 : (3, 9, 2)$ and

(i) L : a straight line from P_1 to P_2 on $z = 2$ plane , **(6 marks)**

(ii) L : a cubic curved path $y = \dfrac{1}{3}x^3$ from P_1 to P_2 on $z = 2$ plane.
Compare this answer with that obtained in *(b)(i)* and comment on the conservative nature of the given vector field. **(7 + 1 marks)**

(iii) Find $\vec{\nabla} \times \vec{F}$. Does this answer in agreement with the comment in *(b)(ii)* ? **(3 + 1 marks)**

Question two

Given a vector field $\vec{F} = \vec{e}_r \left(r^2 \cos(\theta)\right) + \vec{e}_\theta \left(2r^2\right) + \vec{e}_\phi \left(-3\,r^2 \cos(\phi)\right)$ in spherical coordinates ,

(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if $S = S_1 + S_2$ where

$$S_1 : \left(\begin{array}{l} r = 5 \;,\; \frac{\pi}{2} \le \theta \le \pi \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = \vec{e}_r\, r^2 \sin\theta \, d\theta \, d\phi \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{r=5} \vec{e}_r\, 25 \sin\theta \, d\theta \, d\phi \end{array} \right)$$

$$S_2 : \left(\begin{array}{l} \theta = \frac{\pi}{2} \;,\; 0 \le r \le 5 \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = -\,\vec{e}_\theta \, r \sin\theta \, dr \, d\phi \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\theta = \frac{\pi}{2}} -\,\vec{e}_\theta \, r \, dr \, d\phi \end{array} \right)$$

i.e., S is a lower-half semi-spherical closed surface centered at the origin with a radius of 5 . **(10 marks)**

(b) (i) Evaluate $\vec{\nabla} \bullet \vec{F}$ and show that

$$\vec{\nabla} \bullet \vec{F} = 4\,r \cos(\theta) + 2\,r \cot(\theta) + 3\,r \frac{\sin(\phi)}{\sin(\theta)} \quad .$$ **(4 marks)**

(ii) Find the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in *(a)*, i.e.,

$$V : \; 0 \le r \le 5 \;,\; \frac{\pi}{2} \le \theta \le \pi \;,\; 0 \le \phi \le 2\pi \quad \& \quad dv = r^2 \sin\theta \, dr \, d\theta \, d\phi \; .$$

Compare this answer to that obtained in *(a)* and make a brief comment.

(10+1 marks)

Question three

Given the following Bessel's differential equation as :

$$x^2 \frac{d^2 y(x)}{d x^2} + x \frac{d y(x)}{d x} + \left(x^2 - 4\right) y(x) = 0 \quad .$$

Solve by using the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n \, x^{n+s}$ *and* $a_0 \neq 0$.

(a) Write down the indicial equations. Deduce that $s = -2 \ or \ +2$ and $a_1 = 0$.

(9 marks)

(b) Write down the recurrence relation. For $s = -2 \ or \ +2$, set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_6 . Thus write down two independent solution in their power series forms and show that one of the series solutions is a divergent series and the other is linearly dependent to the well-known Bessel's function of the first kind of order 2 , i.e., $J_2(x) \left(\equiv \frac{1}{8} x^2 - \frac{1}{96} x^4 + \frac{1}{3072} x^6 - \frac{1}{184320} x^8 + \cdots\cdots \right)$.

(12+4 marks)

Question four

An elastic string of length *9* is fixed at its two ends, i.e., at $x = 0$ & $x = 9$ and its transverse deflection $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\,\frac{\partial^2 u(x,t)}{\partial x^2} .$$

(a) Set $u(x,t) = F(x)\,G(t)$ and use separation scheme to deduce the following ordinary differential equations :

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = k\,F(x) \\ \dfrac{d^2 G(t)}{d y^2} = 4\,k\,G(t) \end{cases}$$

where k is a separation constant.
In view of the given boundary conditions, k should be a negative constant, i.e., $k < 0$, explain briefly why? **(4+1 marks)**

(b) By direct substitution, show that . $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{9}\right)\cos\left(\frac{2n\pi t}{9}\right)$

where E_n $n = 1,2,3,\cdots\cdots$ are arbitrary constants , satisfies two fixed end conditions , i.e., $u(0,t) = 0 = u(10,t)$, as well as zero initial speed condition , i.e., $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$.

(7 marks)

(c) Then find E_n in terms of n if the initial position of the string , i.e., $u(x,0)$, is given as $u(x,0) = \begin{cases} 2x & if \quad 0 \le x \le 3 \\ -x+9 & if \quad 3 \le x \le 9 \end{cases}$

(hint : $\int_{x=0}^{9} \sin\left(\frac{n\pi x}{9}\right)\sin\left(\frac{m\pi x}{9}\right) d x = \begin{cases} 0 & if \quad n \ne m \\ \frac{9}{2} & if \quad n = m \end{cases}$ &

$$\int x \sin\left(\frac{n\pi x}{9}\right) d x = \frac{81}{n^2 \pi^2} \sin\left(\frac{n\pi x}{9}\right) - \frac{9}{n\pi}\, x \cos\left(\frac{n\pi x}{9}\right) \quad)$$

Thus calculate the values of E_1 , E_2 *and* E_3 . **(10+3 marks)**

Question five

Given the following non-homogeneous differential equation as

$$\frac{d^2\,x(t)}{dt^2} + 6\,\frac{d\,x(t)}{dt} + 13\,x(t) = 10\,e^{-4t} + 15\sin(t) \ ,$$

(a) find its particular solution $x_p(t)$ and show that

$$x_p(t) = 2\,e^{-4t} + \sin(t) - \frac{1}{2}\cos(t) \ .$$ **(9 marks)**

(b) Find the general solution $x_h(t)$ for the homogeneous part of the given differential equation, i.e., $\dfrac{d^2\,x(t)}{dt^2} + 6\,\dfrac{d\,x(t)}{dt} + 13\,x(t) = 0$. **(6 marks)**

(c) Write down the general solution of the given non-homogeneous differential equation $x_g(t)$. If the initial conditions are given as $x(0) = -3$ & $\left.\dfrac{d\,x(t)}{dt}\right|_{t=0} = 5$, find its specific solution $x_s(t)$. **(1+9 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system