UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2017/2018

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	PHY322/P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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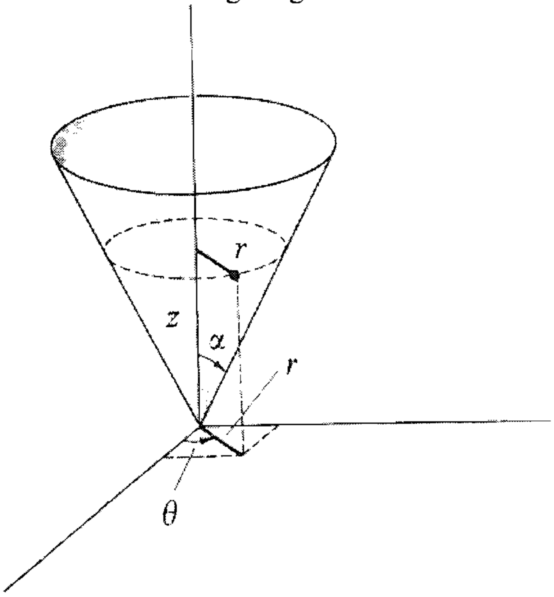
P320 CLASSICAL MECHANICS

Question one

A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α as shown in the following diagram :



The particle is subjected to a gravitational force $\left(-\vec{e}_z\ m\ g\right)$

(a) (i) Write down the Lagrangian for the particle in terms of r & θ and show that

$$L = \frac{1}{2} m \left(\csc^2(\alpha)\,\dot{r}^2 + r^2\,\dot{\theta}^2\right) - m\,g\,r\cot(\alpha) \quad .$$ **(5 marks)**

(Hint : $\vec{v} \equiv \frac{d\vec{s}}{dt}$ and $(d\vec{s})\bullet(d\vec{s}) = (d\,s)^2 = (d\,r)^2 + r^2\,(d\,\theta)^2 + (d\,z)^2$ for cylindrical coordinate system. On the given cone surface one has $z = r\cot(\alpha)$)

(ii) Write down their respective equations of motion for r & θ . **(5 marks)**

(iii) Write down their canonical momenta p_r & p_θ and show that p_θ is a constant. **(2 marks)**

(b) (i) Use the definition of Hamiltonian $H \equiv \sum_{\alpha=1}^{n} \left(p_\alpha\,\dot{q}_\alpha\right) - L$ to deduce that

$$H = \frac{\sin^2(\alpha)\,p_r^{\,2}}{2\,m} + \frac{p_\theta^{\,2}}{2\,m\,r^2} + m\,g\,r\cot(\alpha)$$ **(7 marks)**

(ii) From the Hamiltonian in *(b)(i)*, write down the equations of motion of the system. For each equation obtained here, point out its equivalent equation obtained in *(a)*. **(4+2 marks)**

Question two

For a particle of mass m acted on only by an earth gravitational force of $\vec{F} = -\ \vec{e}_y\ m\ g$ and undergoing a projectile motion near the earth surface in a x-y plane where x-direction is along the horizontal direction.

(a) Write down the Hamiltonian H of the system , i.e., $H(x, y, p_x, p_y)$, and show that

$$H = \frac{p_x^2}{2\,m} + \frac{p_y^2}{2\,m} + m\,g\,y$$

(5 marks)

(b) From the definition of the Poisson brackets , i.e., $[F, G] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right)$,

evaluate $[x, H]$, $[y, H]$, $[p_x, H]$ *and* $[p_y, H]$. **(8 marks)**

(c) For an equation of the type $\frac{d\,u}{d\,t} = [u, H]$ the specific solution of $u(t)$ is given by the following series expansion

$$u(t) = u_0 + [u, H]_0\, t + [[u, H], H]_0 \frac{t^2}{2!} + [[[u, H], H], H]_0 \frac{t^3}{3!} + \cdots\cdots$$

where subscript 0 denotes the initial conditions at $t = 0$.

Use the above relation to show that for the given Hamiltonian, the specific solutions of $x(t)$ and $y(t)$ are given by

$$\begin{cases} x(t) = x_0 + \dfrac{p_{x,0}}{m}\, t \\ y(t) = y_0 + \dfrac{p_{y,0}}{m}\, t - \dfrac{g}{2}\, t^2 \end{cases}$$

where x_0 *and* $p_{x,0}$ are the initial x-position and x-momentum and y_0 *and* $p_{y,0}$ are the initial y-position and y-momentum . **(12 marks)**

Question three

(a) For circular orbits in an attractive central force potential of the form $V = -\frac{k}{r^n}$ where k is a positive constant and $n > 0$, find a relation between the kinetic and potential energies and show that

$$T = \frac{n\,k}{2\,r^n} \quad .$$ **(7 marks)**

(Hint : $\vec{a} = \vec{e}_r\left(\ddot{r} - r\,\dot{\theta}^2\right) + \vec{e}_\theta\left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)$)

(b) An earth satellite moves in an elliptical orbit with period τ, eccentricity ε and semi-major axis a. The maximum radial velocity, named as $v_{\theta,\max}$, occurs at $r = r_{\min}$. Show that

$$v_{\theta,\max} = \frac{2\,\pi\,a}{\tau\,\sqrt{1-\varepsilon^2}}$$ **(7 marks)**

(Hint : $\mu\, r_{\min}\, v_{\theta,\max} = l$, $A = \pi\, a\, b$ *and* $b = a\sqrt{1-\varepsilon^2}$)

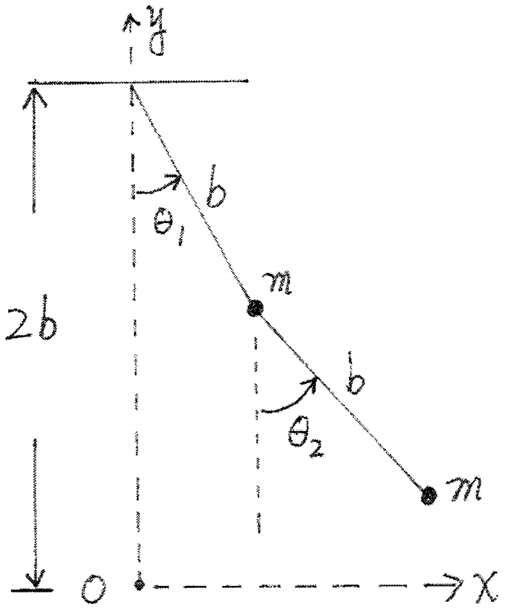
(c) If an earth satellite of 800 kg mass is having a measured velocity of $\vec{v} = \vec{e}_r\, 3000 + \vec{e}_\theta\, 4000$ m/s at the point 600 km directly above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(4 marks)**

(ii) calculate the values of the eccentricity, ε , and show that the orbit is an elliptical orbit. Also calculate its period. **(7 marks)**

Question four

Consider the motion of the bobs in the double pendulum system in the figure below.



Both pendulums are identical and having the length b and bob mass m. The motion of both bobs is restricted to lie in the plane of this paper, i.e., x-y plane.

(a) (i) For small θ_1 *and* θ_2, i.e., $\left(\sin(\theta) \approx \theta \;\; and \;\; \cos(\theta) \approx 1 - \frac{\theta^2}{2} \;\; or \;\; 1\right)$, show that the Lagrangian for the system can be expressed as:

$$L = m\,b^2\,\dot{\theta}_1^2 + \frac{1}{2}m\,b^2\,\dot{\theta}_2^2 + m\,b^2\,\dot{\theta}_1\,\dot{\theta}_2 - m\,g\,b\left(1 + \theta_1^2 + \frac{\theta_2^2}{2}\right) \quad \cdots\cdots \quad (1)$$

where the zero gravitational potential is set at the equilibrium position of the lower bob, i.e., $\theta_1 = 0$, $\theta_2 = 0$ *and* $y = 0$. **(5 marks)**

(ii) Write down the equations of motion and deduce that

$$\begin{cases} 2\,\ddot{\theta}_1 + \ddot{\theta}_2 = -2\frac{g}{b}\theta_1 & \cdots\cdots \quad (2) \\ \ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{b}\theta_2 & \cdots\cdots \quad (3) \end{cases}$$

(5 marks)

(iii) Deduce from eq.(2) & eq.(3) the following :

$$\begin{cases} \ddot{\theta}_1 = -2\frac{g}{b}\theta_1 + \frac{g}{b}\theta_2 & \cdots\cdots \quad (4) \\ \ddot{\theta}_2 = 2\frac{g}{b}\theta_1 - 2\frac{g}{b}\theta_2 & \cdots\cdots \quad (5) \end{cases}$$

(2 marks)

Question four (continued)

(b) (i) Set $\theta_1 = \hat{X}_1 \, e^{i\omega t}$ *and* $\theta_2 = \hat{X}_2 \, e^{i\omega t}$ (where $\hat{X}_1$ *and* $\hat{X}_2$ are constants) and deduce from eq.(4) & eq.(5) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(2\frac{g}{b}\right) & \frac{g}{b} \\ 2\frac{g}{b} & -\left(2\frac{g}{b}\right) \end{pmatrix}$$

(3 marks)

(ii) Find the eigenfrequencies ω of this coupled system and show that they are

$$\omega_1 = \sqrt{\left(2-\sqrt{2}\right)\frac{g}{b}} \quad \& \quad \omega_2 = \sqrt{\left(2+\sqrt{2}\right)\frac{g}{b}}$$

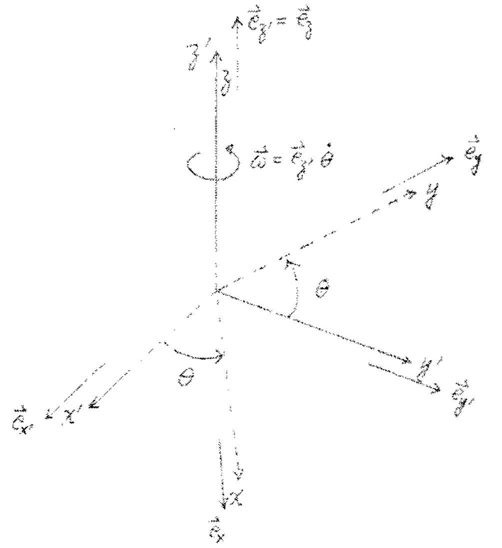
(5 marks)

(iii) Find the eigenvectors corresponding to ω_1 & ω_2 in *(b)(ii)* respectively.

(5 marks)

Question five

(a) Two set of Cartesian coordinate axes are having the same origins and z-axis. The non-prime system (referred to as "rotating" system) is rotating with an angular velocity $\vec{\omega} = \vec{e}_{z'}\,\dot{\theta}$ about the prime system (referred as "fixed" system) as shown below:



For any vector field $\vec{F}$ decomposed into the above two-set of cartesian components, i.e., $\vec{F} = \vec{e}_x\,F_x + \vec{e}_y\,F_y + \vec{e}_z\,F_z = \vec{e}_{x'}\,F_{x'} + \vec{e}_{y'}\,F_{y'} + \vec{e}_{z'}\,F_{z'}$, show that

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \left(\frac{d\vec{F}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{F}$$

where

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \vec{e}_{x'}\,\frac{dF_{x'}}{dt} + \vec{e}_{y'}\,\frac{dF_{y'}}{dt} + \vec{e}_{z'}\,\frac{dF_{z'}}{dt} \quad and$$

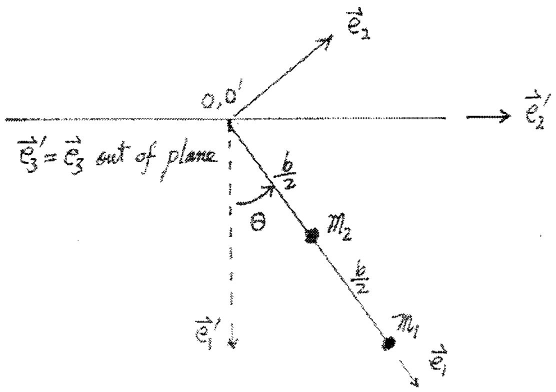
$$\left(\frac{d\vec{F}}{dt}\right)_{rotating} = \vec{e}_x\,\frac{dF_x}{dt} + \vec{e}_y\,\frac{dF_y}{dt} + \vec{e}_z\,\frac{dF_z}{dt}$$

(Hint : $\vec{e}_x = \vec{e}_{x'}\,\cos(\theta) + \vec{e}_{y'}\,\sin(\theta)$, $\vec{e}_y = -\,\vec{e}_{x'}\,\sin(\theta) + \vec{e}_{y'}\,\cos(\theta)$ and $\vec{e}_z = \vec{e}_{z'}$)

(10 marks)

Question five (continued)

(b) A pendulum is composed of a rigid rod of length b with a mass m_1 at its end. Another mass m_2 is placed halfway down the rod. The mass of the rod itself is negligible. Let the fixed and body coordinate systems have their origin at the pendulum pivot point. Let $(\vec{e}_1{}' \ , \ \vec{e}_2{}' \ , \ \vec{e}_3{}')$ *and* $(\vec{e}_1 \ , \ \vec{e}_2 \ , \ \vec{e}_3)$ be the unit vectors of the fixed and body coordinate system respectively as shown below.



(i) Write down the inertia tensor I for the pendulum with respect to the body coordinate system given above and deduce that I is a diagonal matrix with its diagonal elements as $I_{1,1} = 0$ *and* $I_{2,2} = I_{3,3} = \left(m_1 + \frac{m_2}{4} \right) b^2$. **(5 marks)**

(ii) From the equation of rotational motion , i.e., $\dot{\vec{L}} = \vec{N}$ where angular momentum $\vec{L} = I\,\vec{\omega}$ and torque $\vec{N} = \sum_{\alpha} \left(\vec{r}_\alpha \times \vec{F}_\alpha \right)$, deduce the following equation :

$$b^2 \left(m_1 + \frac{m_2}{4} \right) \ddot{\theta} = -\, b\, g \sin(\theta) \left(m_1 + \frac{m_2}{2} \right)$$ **(10 marks)**

(Hint :

$\vec{\omega} = \vec{e}_3{}'\,\dot{\theta}$, $\vec{F}_1 = \vec{e}_1{}'\, m_1\, g$, $\vec{F}_2 = \vec{e}_1{}'\, m_2\, g$,

$\vec{r}_1 = \vec{e}_1{}'\, b\cos(\theta) + \vec{e}_2{}'\, b \sin(\theta)$ *and* $\vec{r}_2 = \vec{e}_1{}' \frac{b}{2}\cos(\theta) + \vec{e}_2{}' \frac{b}{2}\sin(\theta)$)

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad \textit{and reversely} \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \textit{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \textit{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

$$\textit{radius of earth} \quad r_E = 6.4 \times 10^6 \ m$$

$$\textit{mass of earth} \quad m_E = 6 \times 10^{24} \ kg$$

$$\textit{earth attractive potential} \equiv -\frac{k}{r} \quad \textit{where} \quad k = G\, m\, m_E$$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k}} \quad \{(\varepsilon = 0\,,\ circle), (0 < \varepsilon < 1\,,\ ellipse), (\varepsilon = 1\,,\ parabola), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad if \quad m_2 >> m_1$$

$$\textit{For elliptical orbit, i.e., } 0 < \varepsilon < 1\textit{, then} \begin{cases} \textit{semi-major} \quad a = \dfrac{k}{2\,|E|} \\ \textit{semi-minor} \quad b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ \textit{period} \quad \tau = \dfrac{2\,\mu}{l}(\pi\, a\, b) \\ r_{\min} = a\,(1 - \varepsilon) \ \& \ r_{\max} = a\,(1 + \varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r\, (\ddot{r} - r\,\dot{\theta}^2) + \vec{e}_\theta\, (2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,2} & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,1} & -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\,m\,\vec{\omega} \times \vec{v}_r \qquad \textit{where}$$

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$