UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2017/2018

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	PHY322/P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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PHY322/P320 CLASSICAL MECHANICS

Question one

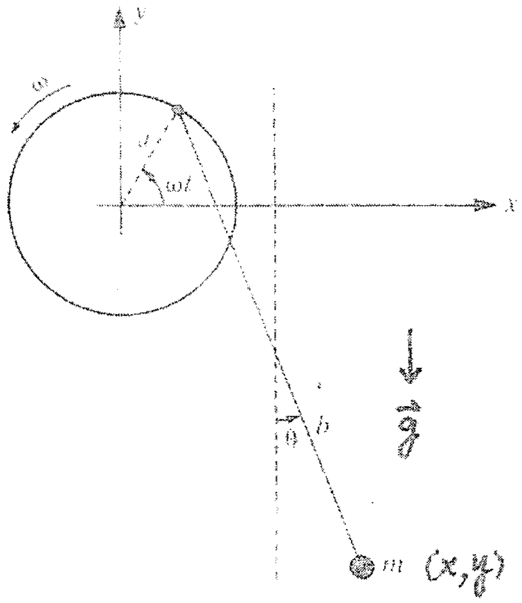
(a) If H denotes the Hamiltonian function and L is the Lagrangian function, use the definition $H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$ (where p_α *and* q_α $(\alpha = 1, 2, \cdots, n)$ are the generalized momenta and coordinates respectively, i.e., $H = H(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$, $L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$, $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$ *and* $\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$) to show that

(i) $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \alpha = 1, 2, \cdots, n$ **(3 marks)**

(ii) $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \quad \alpha = 1, 2, \cdots, n$ **(3 marks)**

(iii) $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ **(6 marks)**

(b) The point of support of a simple pendulum of length b moves on a massless rim of radius a rotating with constant angular velocity ω as shown in the diagram below :



(i) Write down the Lagrangian of the above system in terms of θ and deduce that

$$L = \frac{m}{2}\left(a^2 \omega^2 + b^2 \dot{\theta}^2 + 2\, a\, b\, \omega\, \dot{\theta} \sin(\theta - \omega t)\right) - m\, g\, \left(a \sin(\omega t) - b \cos(\theta)\right)$$

(7 marks)

(ii) Write down the equation of motion for θ and deduce that

$$\ddot{\theta} = \frac{\omega^2 a}{b} \cos(\theta - \omega t) - \frac{g}{b} \sin(\theta)$$ **(6 marks)**

Question two

For a certain dynamical system the kinetic energy T and potential energy V are given by

$T = \dot{q}_1^2 + 2\,\dot{q}_1\,\dot{q}_2 + 3\,\dot{q}_2^2$

$V = 4\,q_1\,q_2$

where q_1 , q_2 are the generalized coordinates.

(a) From the Lagrangian $L \equiv T - V$, find the momentum p_1 & p_2 of the system. **(2 marks)**

(b) Use $H \equiv \left(\sum_{\alpha=1}^{2} p_\alpha\,\dot{q}_\alpha \right) - L$ to find the Hamiltonian function of the system and show that $H = \frac{1}{8}\left(3\,p_1^2 - 2\,p_1\,p_2 + p_2^2\right) + 4\,q_1\,q_2$ **(10 marks)**

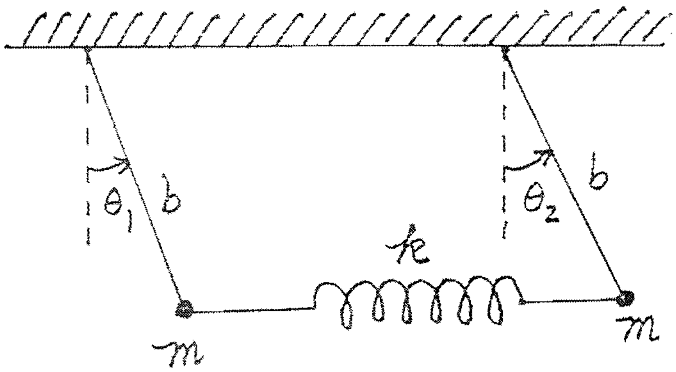
(c) For the Hamiltonian given in (b), write down its Hamiltonian equations of motion. . **(4 marks)**

(d) From the definition of the Poisson brackets , i.e., $[F,G] \equiv \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha} \right)$, evaluate $[q_1, H]$, $[q_2, H]$, $[p_1, H]$ *and* $[p_2, H]$. **(8 marks)**

(e) Compare the results obtained in (c) to those obtained in (d) and make a brief comment. **(1 mark)**

Question three

Two pendulums of equal lengths b and equal masses m are connected by a spring of force constant k as shown below. The spring is unstretched in the equilibrium position, i.e., $\theta_1 = 0 \;\; and \;\; \theta_2 = 0$.



(i) For small $\theta_1 \;\; and \;\; \theta_2$, i.e.,

$$\left(\sin(\theta_1) \approx \theta_1 \;\; , \;\; \sin(\theta_2) \approx \theta_2 \;\; , \;\; \cos(\theta_1) \approx 1 - \frac{\theta_1^2}{2} \;\; and \;\; \cos(\theta_2) \approx 1 - \frac{\theta_2^2}{2} \right)$$

, show that the Lagrangian for the system can be expressed as:

$$L = \frac{1}{2} m b^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \right) - \frac{m g b}{2} \left(\theta_1^2 + \theta_2^2 \right) - \frac{k b^2}{2} \left(\theta_1 - \theta_2 \right)^2$$

where the zero gravitational potential is set at the equilibrium position. **(6 marks)**

(ii) Write down the equations of motion and deduce that

$$\begin{cases} \ddot{\theta}_1 = -\left(\dfrac{m g + k b}{m b} \right) \theta_1 + \dfrac{k}{m} \theta_2 \\ \ddot{\theta}_2 = \dfrac{k}{m} \theta_1 - \left(\dfrac{m g + k b}{m b} \right) \theta_2 \end{cases}$$

(6 marks)

(iii) Set $\theta_1 = \hat{X}_1 \, e^{i \omega t} \;\; and \;\; \theta_2 = \hat{X}_2 \, e^{i \omega t}$ (where $\hat{X}_1 \;\; and \;\; \hat{X}_2$ are constants) and deduce from the equations in (ii) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(\dfrac{m g + k b}{m} \right) & \dfrac{k b}{m} \\ \dfrac{k b}{m} & -\left(\dfrac{m g + k b}{m} \right) \end{pmatrix}$$

(3 marks)

(iv) Find the eigenfrequencies ω of this coupled system and show that they are

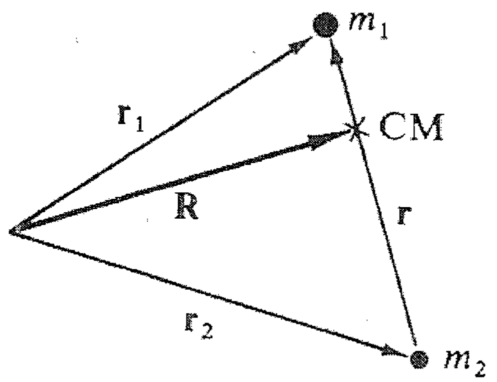
$$\omega_1 = \sqrt{\frac{g}{b}} \qquad \& \qquad \omega_2 = \sqrt{\frac{m g + 2 k b}{m b}}$$

(6 marks)

(v) Find the eigenvectors for $\omega_1 \;\; and \;\; \omega_2$ respectively. **(4 marks)**

Question four

(a) A two-body system is depicted below



where $\vec{r}_1$ & $\vec{r}_2$ are the position vectors of m_1 & m_2 respectively.
Define the center of mass of the system and show that the total kinetic energy of the system, i.e., $T = \frac{1}{2} m_1 \left(\dot{\vec{r}}_1 \bullet \dot{\vec{r}}_1\right) + \frac{1}{2} m_2 \left(\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2\right)$, can be reduced to

$T = \frac{1}{2} \mu \left(\dot{\vec{r}} \bullet \dot{\vec{r}}\right) \left(\textit{where} \quad \mu = \frac{m_1\, m_2}{m_1 + m_2} \textit{ is the reduced mass} \right)$ if the center of mass is chosen to be the origin. **(1+6 marks)**

(b) Starting from the law of conservation of angular momentum l, derive Kepler's third law, i.e., the relation between the period τ of a closed orbit in an attractive inverse square central force and the area A of the orbit. Show that

$\tau = \frac{2\mu}{l} A$ where μ is the reduced mass of the system. **(7 marks)**

(c) If an earth satellite of 500 kg mass is having a pure tangential speed $v_\theta = 8{,}000$ m/s at its near-earth-point 600 km above the earth surface,

(i) calculate the values of l and E of this satellite, **(3 marks)**

(ii) calculate the values of the eccentricity, ε, and show that the orbit is an elliptical orbit. Also calculate its period. **(6 marks)**

(iii) determine the value of the v_θ at the same given near-earth-point such that the satellite orbit is a circular orbit, **(2 marks)**

(Hint : $E = \frac{1}{2}\mu v_\theta^2 - \frac{k}{r} \xrightarrow{\text{circular orbit}} -\frac{k}{2r}$)

Question five

(a) (i) Find the horizontal deflection d resulting from the Coriolis force $\left(-2\,m\,\vec{\omega}\times\vec{v}_r\right)$ of a particle falling freely from a height h to the ground at a northern latitude λ and show that

$$d \approx \frac{1}{3}\,\omega\cos(\lambda)\sqrt{\frac{8h^3}{g}}$$ **(11 marks)**

(Hint : Chose $\vec{e}_x$ *due south* , $\vec{e}_y$ *due east* & $\vec{e}_z$ *upward* on earth surface, then $\vec{a}_{eff} \approx -\vec{e}_z\, g - 2\,\vec{\omega}\times\vec{v}_r$, $\vec{v}_r \approx \vec{e}_z\,(-g\,t)$, $\vec{\omega} = \vec{e}_x\,(-\omega\cos(\lambda)) + \vec{e}_z\,(\omega\sin(\lambda))$)

(ii) Given the values of $\lambda = 60^0$, $h = 100$ m and $\omega = 2\pi$ rad/day (i.e., $\omega = 7.27\times10^{-5}$ rad/s) , determine the value of the deviation distance d .

(2 marks)

(b) By proper choice of the body coordinate system for a well-shaped rigid body, its inertial tensor can be a diagonal matrix with I_1 , I_2 & I_3 diagonal elements. Its force-free pure-rotational motion obeys the following Euler's equations :

$$\begin{cases} (I_2 - I_3)\,\omega_2\,\omega_3 - I_1\,\dot{\omega}_1 = 0 & \cdots\cdots \quad (1) \\ (I_3 - I_1)\,\omega_3\,\omega_1 - I_2\,\dot{\omega}_2 = 0 & \cdots\cdots \quad (2) \\ (I_1 - I_2)\,\omega_1\,\omega_2 - I_3\,\dot{\omega}_3 = 0 & \cdots\cdots \quad (3) \end{cases}$$

(i) In the case of $I_1 = I_2 \xrightarrow{set\ as} I_0$ & $I_3 = 2\,I_0$, deduce from the above Euler's equations that

$$\begin{cases} \omega_3 = const. \xrightarrow{set\ as} K & \cdots\cdots \quad (4) \\ \dot{\omega}_1 = -K\ \omega_2 & \cdots\cdots \quad (5) \\ \dot{\omega}_2 = K\,\omega_1 & \cdots\cdots \quad (6) \end{cases}$$ **(4 marks)**

(ii) Deduce from *eq.(5)* and *eq.(6)* in (b)(i) that

$\ddot{\omega}_1 = -K^2\,\omega_1 \quad \cdots\cdots \quad (7)$ **(2 marks)**

(iii) By direct substitution, show that $\omega_1 = A\cos(K\,t + B)$ is the solution to *eq.(7)* with A & B constant values linking to the given initial value of $\vec{\omega}$. **(2 marks)**

(iv) Substitute $\omega_1 = A\cos(K\,t + B)$ into *eq.(5)* and deduce that

$\omega_2 = A\sin(K\,t + B)$. **(2 marks)**

(v) Show that the magnitude of $\vec{\omega}$ is constant for all time t . **(2 marks)**

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad \text{and reversely} \quad \vec{F} = -\vec{\nabla} V$$

$$L = T - V = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \quad \text{and} \quad \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_{\alpha=1}^{n} (p_\alpha \, \dot{q}_\alpha) - L = H(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u, v] \equiv \sum_{\alpha=1}^{n} \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11} \ \frac{N\, m^2}{kg^2}$$

radius of earth $r_E = 6.4 \times 10^6 \ m$

mass of earth $m_E = 6 \times 10^{24} \ kg$

earth attractive potential $\equiv -\frac{k}{r}$ *where* $k = G\, m\, m_E$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k^2}} \quad \{(\varepsilon = 0, \ circle), (0 < \varepsilon < 1, \ ellipse), (\varepsilon = 1, \ parabola), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_2 >> m_1$$

$$\text{For elliptical orbit, i.e., } 0 < \varepsilon < 1, \text{ then} \begin{cases} semi-major \ \ a = \dfrac{k}{2|E|} \\ semi-minor \ \ b = \dfrac{l}{\sqrt{2\,\mu\,|E|}} \\ period \ \ \tau = \dfrac{2\,\mu}{l}(\pi\, a\, b) \\ r_{\min} = a(1-\varepsilon) \ \& \ r_{\max} = a(1+\varepsilon) \end{cases}$$

for plane polar (r, θ) *system with unit vectors* $(\vec{e}_r, \vec{e}_\theta)$, *we have*

$$\begin{cases} \vec{v} = \vec{e}_r\, \dot{r} + \vec{e}_\theta\, r\, \dot{\theta} \\ \vec{a} = \vec{e}_r\left(\ddot{r} - r\,\dot{\theta}^2\right) + \vec{e}_\theta\left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right) \end{cases}$$

$$\vec{\nabla} f = \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta}$$

Useful informations (continued)

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,1} \, x_{\alpha,2} & -\sum_\alpha m_\alpha \, x_{\alpha,1} \, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,2} \, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha \, x_{\alpha,2} \, x_{\alpha,3} \\ -\sum_\alpha m_\alpha \, x_{\alpha,3} \, x_{\alpha,1} & -\sum_\alpha m_\alpha \, x_{\alpha,3} \, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) - 2\,m\,\vec{\omega} \times \vec{v}_r$ *where*

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$