UNIVESITY OF SWAZILAND
FACULTY OF SCIENCE AND ENGINEERING
DEPARTMENT OF PHYSICS

Main Examination 2017/2018
COURSE NAME: Quantum Mechanics I
COURSE CODE: PHY341/PHY342
TIME ALLOWED: 3 hours

ANSWER ANY FOUR QUESTIONS. ALL QUESTIONS CARRY EQUAL (**25**) MARKS

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Question 1

(a). Explain what was learned about quantization of radiation or a mechanical system from the following experiments,

(i). The photo-electric effect. [**5 marks**]

(ii). Franck-Hertz experiment. [**5 marks**]

(iii). Compton scattering. [**5 marks**]

(b). Show that the expectation value of $\hat{p}$ obeys Newton's second law. [**6 marks**]

(*Hint: Prove Ehrenfest's thoerem that* $\frac{d\langle\hat{p}\rangle}{dt} = \langle -\frac{dV}{dx} \rangle$)

(c). Show that the matrix $\mathbf{T} = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$ is Hermitian. [**4 marks**]

Question 2

(a). State, with justification, whether each of the following sets of quantum numbers could be used to label a possible wave-function of the hydrogen atom. [**5 marks**]

(i). $n = 1, \ell = 1, m_\ell = 1, m_s = \frac{1}{2}$

(ii). $n = 3, \ell = 1, m_\ell = 2, m_s = 0$

(iii). $n = 4, \ell = 1, m_\ell = -1, m_s = \frac{-1}{2}$

(iv). $n = 2, \ell = 1, m_\ell = 0, m_s = 0$

(v). $n = 4, \ell = 3, m_\ell = -1, m_s = \frac{1}{2}$

(b). The following wave functions are energy eigenfunctions of the hydrogen atom.

$$\Psi_1(r,\theta,\phi) = \frac{1}{\sqrt{32\pi a_0^2}}\left(2 - \frac{r}{a_0}\right)e^{\frac{-r}{2a_0}}$$

$$\Psi_2(r,\theta,\phi) = \frac{1}{81\sqrt{\pi a_0^3}}\left(\frac{r}{a_0}\right)^2 e^{\frac{-r}{3a_0}} \sin\theta \cos\theta e^{-i\phi}$$

(i). Deduce the quantum numbers n, ℓ, and m_ℓ for each wavefunction. [**10 marks**]

(ii). Verify if $\Psi_1(r,\theta,\phi)$ is normalised and calculate the expectation value of the electron-nuclear separation in the hydrogen atom for this wavefunction [**10 marks**]

Question 3

(a). A bead of mass m is constrained to move freely on a circular hoop of radius R a shown in fig. 1. For such a system, the momentum operator is defined as $\hat{p}_\varphi = i\frac{\hbar}{R}\frac{\partial}{\partial\varphi}$, where φ is the angle measured counter-clockwise as shown. The eigenfunctions of $\hat{p}$ are $f_p(\varphi)$ and the obey the periodic boundary condition $f(0) = f(2\pi)$. These eigen-functions form a Hilbert space.

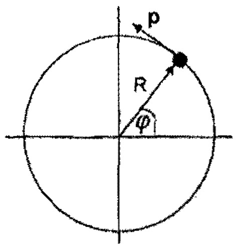


Figure 1:

(i). Prove that $\hat{p}_\varphi$ is a Hermitian observable from this Hilbert space. [**5 marks**]

(ii). Solve the eigen-value equation $\hat{p}f_p = pf_p$, in terms of the eigen-value p and the angle φ. [**5 marks**]

(b). Electrons are passing through two slits that are 100 nm apart. Use Heisenberg's uncertainty principle to find the minimum spread in the electron's momentum in the direction parallel to the plane of the slits. [**3 marks**]

(c). A particle has a time-dependant wave-function, $\Psi(x, t)$.

(i). What is the physicsl interpretation of $|\Psi|^2$? [**2 marks**]

(ii). What is the physical interpretation of $\langle\Psi| \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) |\Psi\rangle$? [**3 marks**]

(d). (i). Write down the time-independant Schrödinger equation for a 1-dimensional Harmonic oscillator, for a particle of mass m osscilating with frequency ω. [**2 marks**] (ii). The ground state wavefunction for the above equation is given by $\psi_0(x) = \frac{1}{\sqrt{\pi^{1/2}x_0}} e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}$, and it has energy $\frac{\hbar\omega}{2}$. Show explicitly that $\psi_0(x)$ is an eigenstate and find the natural length scale, x_0, in terms of m, ω and $\hbar$. **[5 marks]**

Question 4

(a). A infinite cubical well ("or particle in a box") has a potential, defined in Cartesian 3D coordinates as

$$V(x, y, z) = \begin{cases} 0, & \text{if } x, y, z \text{ are all between 0 and a} \\ \infty, & \text{otherwise} \end{cases}$$

(i). What is the time-dependant Schrödinger equation for the particle inside the box ? **[5 marks]**

(ii). Use the separation of variables ansatz ($\Psi(x, y, z) = X(x)Y(y)Z(z)$) to derive **three** separate ordinary differential equations for X, Y and Z in terms of K_x, K_y and K_z, where $E = \frac{\hbar^2}{2m}(K_x^2 + K_y^2 + K_z^2)$. **[5 marks]**

(iii). Find solutions to the equations above. Apply the boundary conditions to show that

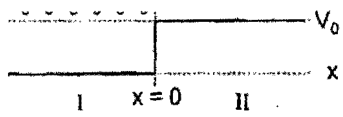
$$K_x a = n_x \pi$$
$$K_y a = n_y \pi$$
$$K_z a = n_z \pi$$

where $n_x, n_y, n_z = 1, 2, 3, ...$ **[5 marks]**

(b). Consider a step potential;

$$V(x) = \begin{cases} 0, & \text{if } x < 0 \\ V_0, & \text{if } x \geq 0 \end{cases}.$$

Consider the initial condition where a single particle of energy $E > V_0$ is incident from the left and no particle is incident from the right. Use $k = \sqrt{\frac{2mE}{\hbar}}$ and $l = \sqrt{\frac{2m(E - V_0)}{\hbar}}$ to show that the time-independant



Schrödinger equation for region I and II are

$$\frac{\partial^2 \Psi_I(x)}{\partial x^2} + k^2 \Psi_I(x) = 0$$
$$\frac{\partial^2 \Psi_{II}(x)}{\partial x^2} + l^2 \Psi_{II}(x) = 0.$$

[**10 marks**]

Question 5

(a). How did the Stern-Gerlach experiment demonstrate that a measurement in quantum mechanics affects the system being measured? [**2 marks**]

(b). A particle is in a region of space where it has a wave function given by

$$\Psi(x,t) = \begin{cases} 0, & x < 0 \\ Ae^{-kx} e^{\left\{\frac{iEt}{\hbar}\right\}}, & x > 0 \end{cases}$$

What is the value of A? [**3 marks**]

(c). A particle of mass m is confined to a harmonic oscillator potential given by $V = \dfrac{mx^2\omega^2}{2}$, where $\omega^2 = \dfrac{k}{m}$ and k is the spring constant. The particle is in a state defined by the wave function

$$\Psi(x,t) = Ae^{\left(\frac{-mx^2\omega}{2\hbar} - \frac{i\omega t}{2}\right)}.$$

Show that this is a solution to the Schrödinger equation. [**6 marks**]

(d). A particle moves in an infinite potential well described by

$$V(x) = \begin{cases} 0, & |x| > \frac{a}{2} \\ \infty, & |x| \le \frac{a}{2} \end{cases}$$

The eigenfunctions are of the form $\psi_n(x) = A_n \cos(k_n x)$ or $\psi_n(x) = B_n \sin(k_n x)$ depending on the value of n. For $n = 3$,

$$\psi_3(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{3\pi x}{a}\right), & |x| \leq \frac{a}{2} \\ 0, & |x| > \frac{a}{2} \end{cases}$$

(i). What are the expectation values of x and x^2 in the $n = 3$ state? **[7 marks]**

(ii). What are the expectation values of p and p^2 in the $n = 3$ state? **[7 marks]**

Appendix

Some useful information:

1
$$\int x^2 \sin(ax)dx = \frac{2x}{a^2}\sin(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos(ax)$$

2
$$\int x^2 \cos(ax)dx = \frac{2x}{a^2}\cos(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\sin(ax)$$

3
$$\int x^2 \cos^2(bx)dx = \frac{4b^3x + 3(2b^2x^2 - 1)\sin(2bx) + 6bx\cos(2bx)}{24b^3}$$

4
$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

5 $\int_0^\infty x^n e^{\frac{-x}{a}} dx = n! a^{n+1}$ for any non-negative integer, n.

6 Planck's constant $h = 6.663 \times 10^{-34} J \cdot s$

7 Dirac's constant $\hbar = 1.05 \times 10^{-31} J \cdot s$

8 Permittivity of vaccum $\epsilon_0 = 8.854 \times 10^{-12} C^2 N^{-1} m^{-2}$

9 Ground state hydrogen energy $-E_1 = 13.6057 eV$

10 Bohr energies $E_n = \frac{E_1}{n^2}$

11 Hydrogen ground state wavefunction $\Psi_0 = \frac{1}{\sqrt{\pi a^3}} e^{\frac{-r}{a}}$

Y_0^0	$\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$
Y_1^0	$\left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$
$Y_1^{\pm}$	$\mp \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta e^{\pm i\phi}$

Table 1: The first few spherical harmonics, $Y_l^m(\theta, \phi)$

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R_{10}	$2a^{\frac{-3}{2}} e^{\frac{-r}{a}}$
R_{20}	$\frac{1}{\sqrt{2}} a^{\frac{-3}{2}} \left(1 - \frac{r}{2a}\right) e^{\frac{-r}{2a}}$
R_{21}	$\frac{1}{\sqrt{24}} a^{\frac{-3}{2}} \left(\frac{r}{2}\right) e^{\frac{-r}{2a}}$

Table 2: The first hydrogen radial wavefunctions, $R_{nl}(r)$

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