UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

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TITLE OF PAPER	**:**	**MATHEMATICAL METHODS FOR PHYSICISTS**
COURSE NUMBER	**:**	**P272/PHY271**
TIME ALLOWED	**:**	**THREE HOURS**
INSTRUCTIONS	**:**	**ANSWER ANY <u>FOUR</u> OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES <u>25</u> MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

THIS PAPER HAS <u>EIGHT</u> PAGES, INCLUDING THIS PAGE.

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Question one

Given a vector field $\vec{F} = \vec{e}_r\left(r^2\right) + \vec{e}_\theta\left(6r^2 \sin\theta\right) + \vec{e}_\phi\left(3\,r^2 \cos\phi\right)$ in spherical coordinates ,

(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if $S = S_1 + S_2$ where

$$S_1 : \left(r = 4 \; , \; 0 \le \theta \le \frac{\pi}{2} \; , \; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = \vec{e}_r \, r^2 \sin\theta \, d\theta \, d\phi \xrightarrow{r=4} \vec{e}_r \, 16 \sin\theta \, d\theta \, d\phi \right)$$

$$S_2 : \left(\theta = \frac{\pi}{2} \; , \; 0 \le r \le 4 \; , \; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = \vec{e}_\theta \, r \sin\theta \, dr \, d\phi \xrightarrow{\theta = \frac{\pi}{2}} \vec{e}_\theta \, r \, dr \, d\phi \right)$$

i.e., S is a upper-half semi-spherical closed surface centered at the origin with a radius of 4 . **(10 marks)**

(b) (i) Evaluate $\vec{\nabla} \bullet \vec{F}$ and show that

$$\vec{\nabla} \bullet \vec{F} = 4\,r + 12\,r\cos(\theta) - \frac{3\,r\sin(\phi)}{\sin(\theta)} \quad .$$ **(4 marks)**

(ii) Find the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in *(a)*, i.e.,

$$V : \; 0 \le r \le 4 \; , \; 0 \le \theta \le \frac{\pi}{2} \; , \; 0 \le \phi \le 2\pi \quad \& \quad dv = r^2 \sin\theta \, dr \, d\theta \, d\phi \; .$$

Compare this answer to that obtained in *(a)* and make a brief comment. **(10+1 marks)**

Question two

(a) Given the following Laplace equation in cylindrical coordinates as :

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f(\rho,\phi,z)}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f(\rho,\phi,z)}{\partial \phi^2}+\frac{\partial^2 f(\rho,\phi,z)}{\partial z^2}=0$$

utilize the separation of variable scheme to split it into three ordinary differential equations.

(7 marks)

(b) Given the following Legendre's differential equation as :

$$\left(1-x^2\right)\frac{d^2 y(x)}{d x^2}-2x\frac{d y(x)}{d x}+20\, y(x)=0 \ .$$

Solve by using the power series method , i.e., setting $y(x)=\sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$

(i) Write down the indicial equations and recurrence relation. Deduce that $s=0$ *or* 1 and $a_1=0$.

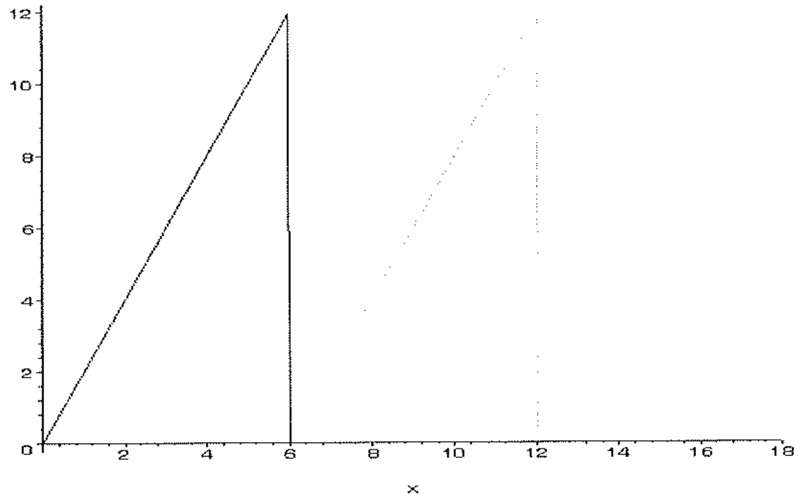
(8 marks)

(ii) For $s=0$ case, set $a_0=1$ and use the recurrence relation to calculate the values of a_n up to the value of a_8 . Then write down the independent solution in its power series form and show that it's a polynomial and linearly dependent to the well-known Legendre polynomial of order 4 ,i.e., $P_4(x)\left(\equiv \frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}\right)$.

(8+2 marks)

Question three

Given the following non-homogeneous differential equation as $\frac{d^2 x(t)}{dt^2} + \frac{1}{9} x(t) = f(t)$, where $f(t)$ is a periodic jigsaw shape driving force of period 6 , i.e., $f(t) = f(t+6) = f(t+12) = \cdots\cdots$ and plotted against its first three periods as shown below :



i.e., its first period description is $f(t) = 2\,t \quad for \quad 0 \le t \le 6$

(a) Find the Fourier series expansion of $f(t)$ and show that

$$f(t) = 6 + \sum_{n=1}^{\infty} \left(-\frac{12}{n\pi} \right) \sin\left(\frac{n\pi t}{3} \right) \quad \cdots\cdots \quad (1)$$

(9 marks)

(b) Find its particular solution $x_p(t)$ corresponding to the given $f(t)$ and show that

$$x_p(t) = 54 + \sum_{n=1}^{\infty} \frac{108}{n\pi\left(n^2 \pi^2 - 1\right)} \sin\left(\frac{n\pi t}{3} \right)$$

(12 marks)

(c) for the homogeneous part of the given non-homogeneous differential equation , i.e., $\frac{d^2 x(t)}{dt^2} + \frac{1}{9} x(t) = 0$, set $x(t) = e^{\alpha t}$ and find the appropriate values of α and thus write down its general solution $x_h(t)$. Then write down the general solution of the given non-homogeneous differential equation $x_g(t)$. **(3+1 Marks)**

Question four

Given the following non-homogeneous differential equation as
$y''(t) - 3\,y'(t) + 2\,y(t) = h(t) \quad \& \quad h(t) = 2\cos(3t)$
and also given the following initial condition that $y(0) = 2 \;\; \& \;\; y'(0) = 1$,

(a) Find the Laplace transform of $y(t)$, i.e., $L\{y(t)\} \xrightarrow{\text{set as}} Y(s)$, from the above given non-homogeneous differential equation and initial condition, deduce that
$Y(s) = F(s) + G(s)\,H(s)$ *where*

$$\begin{cases} F(s) = \dfrac{(2\,s - 5)}{(s^2 - 3\,s + 2)} \\ G(s) = \dfrac{1}{(s^2 - 3\,s + 2)} \\ H(s) = \dfrac{2\,s}{s^2 + 9} \qquad \left(Note: \;\; H(s) = L\{h(t)\} \;\leftrightarrow\; h(t) = L^{-1}\{H(s)\}\right) \end{cases}$$

(9 marks)

(b) (i) Find the inverse Laplace transform of $G(s)$, i.e., $L^{-1}\{G(s)\}$, and name it as $g(t)$. Show that $g(t) = -\,e^{t} + e^{2t}$
(Hint : convert $G(s)$ into its partial fractions and then use the given conversion table to find the answers.)

(7 marks)

(ii) Find $L^{-1}\{G(s)\,H(s)\}$ by using convolution theory , i.e.,
$L^{-1}\{G(s)\,H(s)\} = (g * h)(t) = \int_{\tau=0}^{t} g(t-\tau)\,h(\tau)\,d\tau$, and show that

$$L^{-1}\{G(s)\,H(s)\} = -\frac{1}{5}e^{t} + \frac{4}{13}e^{2t} - \frac{7}{65}\cos(3t) - \frac{9}{65}\sin(3t)$$

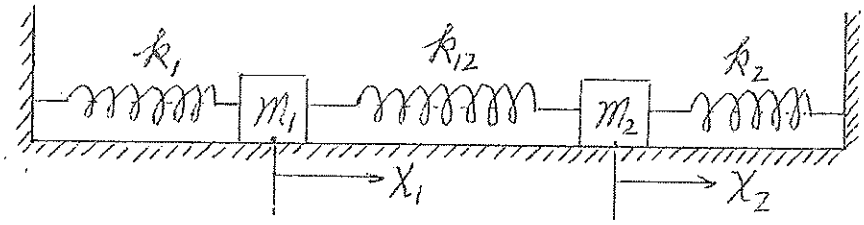
(Hint : $\int e^{a\tau}\cos(b\,\tau)\,d\tau = \dfrac{b}{a^2 + b^2}\,e^{a\tau}\sin(b\,\tau) + \dfrac{a}{a^2 + b^2}\,e^{a\tau}\cos(b\,\tau)$)

(7 marks)

(iii) Assuming $L^{-1}\{F(s)\}$ was found by similar procedures as done in *(b)(i)* as $L^{-1}\{F(s)\} = 3\,e^{t} - e^{2t}$, then write down the specific solution of $y(t)$, i.e., $L^{-1}\{Y(s)\}$, and simplify it. **(2 marks)**

Question five

Two simple harmonic oscillators are joined by a spring with a spring constant k_{12} as shown in the diagram below :



The equations of motion for this coupled oscillator system ignoring friction are given as

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -(k_1 + k_{12})\, x_1(t) + k_{12}\, x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = k_{12}\, x_1(t) - (k_2 + k_{12})\, x_2(t) \end{cases}$$

where x_1 & x_2 are horizontal displacements of m_1 & m_2 measured from their respective resting positions.

If given $m_1 = 1\ kg$, $m_2 = 2\ kg$, $k_1 = 3\ \dfrac{N}{m}$, $k_2 = 6\ \dfrac{N}{m}$ & $k_{12} = 6\ \dfrac{N}{m}$.

(a) Set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$. Then the above given equations can be deduced to the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -9 & 6 \\ 3 & -6 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} .$$ **(5 marks)**

(b) Find the eigenfrequencies ω of the given coupled system . **(5 marks)**

(c) Find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in *(b)* . **(5 marks)**

(d) Write down the general solutions for $x_1(t)$ & $x_2(t)$. **(2 marks)**

(e) Find the specific solutions for $x_1(t)$ & $x_2(t)$ if the initial conditions are given as

$$x_1(0) = 1\ ,\ x_2(0) = 0\ ,\ \left.\frac{d\,x_1(t)}{d\,t}\right|_{t=0} = -1\ \ \&\ \ \left.\frac{d\,x_2(t)}{d\,t}\right|_{t=0} = 0\ .$$ **(8 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$f(t) = f(t+2L) = f(t+4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \textit{where}$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \quad \textit{for} \quad n = 1,2,3,\cdots$$

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$

$F(s) \equiv L\{f(t)\}$	$f(t)$
$1/s$ $1/s^2$ $1/s^n \quad n=1,2,3,\cdots$	1 t $t^{n-1}/(n-1)! \quad n=1,2,3,\cdots$
$1/(s-a)$ $1/(s-a)^2$ $1/(s-a)^n \quad n=1,2,3,\cdots$ $F(s-a)$	e^{at} $e^{at}\, t$ $e^{at}\, t^{n-1}/(n-1)! \quad n=1,2,3,\cdots$ $e^{at}\, f(t)$ *s – shift theory*
$1/(s^2+\omega^2)$ $s/(s^2+\omega^2)$ $1/(s^2-a^2)$ $s/(s^2-a^2)$	$\sin(\omega t)/\omega$ $\cos(\omega t)$ $\sinh(at)/a$ $\cosh(at)$
$e^{-as}/s \quad a>0$ $e^{-as}\, F(s) \quad a>0$	$u(t-a) \quad a>0$ $f(t-a)\, u(t-a)$ *t – shift theory*
$F(s)\, G(s)$	$h(t)=(f*g)(t)=\int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$ $=\int_{\tau=0}^{t} f(t-\tau)\, g(\tau)\, d\tau$ *convolution theory*

where $u(t-a) \equiv \begin{cases} 0 & \text{if} \quad t<a \\ 1 & \text{if} \quad t>a \end{cases}$ *is an unitary step function & its Laplace transform is*

$$L\{u(t-a)\} = \int_{t=0}^{\infty} u(t-a)\, e^{-st}\, dt \quad \text{where} \quad s>0$$

$$= \int_{t=0}^{a} (0)\, e^{-st}\, dt + \int_{t=a}^{\infty} (1)\, e^{-st}\, dt$$

$$= 0 + \left(-\frac{1}{s}\, e^{-st}\right)\Bigg|_{t=a}^{\infty} = \left(-\frac{1}{s}\, e^{-st}\right)\Bigg|_{t=0}^{\infty} = (0) - \left(-\frac{1}{s}\, e^{-sa}\right) = \frac{1}{s}\, e^{-sa}$$

$$L\{y'(t)\} = -\, y(0) + s\, L\{y(t)\}$$

$$L\{y''(t)\} = -\, y'(0) - s\, y(0) + s^2\, L\{y(t)\}$$