UNIVERSITY OF ESWATINI

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION: 2019/2020

TITLE OF THE PAPER: MATHEMATICAL PHYSICS

COURSE NUMBER: PHY271

TIME ALLOWED: THREE HOURS

INSTRUCTIONS: ANSWER ANY **FOUR** OUT OF THE FIVE QUESTIONS. EACH QUESTION CARRIES 25 POINTS. POINTS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THE PAPER HAS 5 PAGES, INCLUDING THIS PAGE.

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Question 1

(a) Calculate the square-root of the imaginary number $-i$:

$$\sqrt{-i}.$$

[4 marks]

(b) Given $f(z) = z^2 + z + 1$, evaluate $f(2+2i)$.

[4 marks]

(c) An electron moving in a thin wire has a wave function

$$U(x) = \frac{c}{x+i}$$

where x is the coordinate of the electron and c is a positive constant.

(i) Calculate and sketch the probability function $p(x) = U^*U$ of the particle (U^* is the complex conjugate of U).

[5 marks]

(ii) Where is the maximum of this probability?

[2 marks]

(d) In integral tables you can find the integrals for such functions as

$$\int dx e^{ax} \cos bx, \quad \text{or} \quad \int dx e^{ax} \sin bx.$$

Show how easy it is to do these by applying Euler identity $e^{ibx} = \cos bx + i \sin bx$ at once.

[10 marks]

Question 2

(a) Solve the following differential equation

$$y'' + 2y' - 15y = 0, \quad y = -1, y' = 1, \text{ at } x = 0.$$

[9 marks]

(b) Consider the differential equation

$$y'' - 3y' + 2y = f(x)$$

What would you try for the particular solution if $f(x) =$

(i) x^2,
(ii) $x\sin(x)$,
(iii) $\sinh x$, and
(iv) $e^{2x} + \cos^2 x$?

[16 marks]

Question 3

(a) For what values of α are the vectors $\mathbf{A} = \alpha\hat{x} - 2\hat{y} + \hat{z}$ and $\mathbf{B} = 2\alpha\hat{x} + \alpha\hat{y} - 4\hat{z}$ orthogonal?

[6 marks]

(b) On the interval $0 < x < L$ with a scalar product defined as $\langle f, g\rangle = \int_0^L dx\, f(x)^* g(x)$, show that these are zero making the functions orthogonal:

(i) x and $L - 3x/2$,
(ii) $\sin \pi x/L$ and $\cos \pi x/L$,
(iii) $\sin 3\pi x/L$ and $L - 2x$.

[9 marks]

(c) In a given basis, an operator has the values

$$A(\hat{e}_1) = \hat{e}_1 + 3\hat{e}_2 \text{ and } A(\hat{e}_2) = 4\hat{e}_1 + 2\hat{e}_2.$$

(i) Express A in a matrix form.
(ii) Find the eigenvalues and eigenvectors of A.

[10 marks]

Question 4

(a) Use the method of separation of variables to obtain the solution $u(x,t)$ for the one-dimensional diffusion equation

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

with the boundary condition that $u(x, t \to \infty) = 0$ for all x.

[9 marks]

(b) Evaluate the line integral

$$I = \int_C \mathbf{a} \cdot d\mathbf{r}, \text{ where } \mathbf{a} = (x+y)\hat{x} + (y-x)\hat{y}$$

along each of the following paths in the xy-plane

(i) the parabola $y^2 = x$ from (1,1) to (4,2).

[5 marks]

(ii) the line $y = 1$ from (1,1) to (4,1), followed by the line $x = 4$ from (4,1) to (4,2).

[5 marks]

(c) A small particle of mass m orbits a much larger mass M centered at the origin. According the Newton's law of gravitation, the position vector $\mathbf{r}$ of the small mass obeys the differential equation

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^2}\hat{r}.$$

Show that the vector $\mathbf{r} \times d\mathbf{r}/dt$ is a constant of motion

[6 marks]

Question 5

(a) Evaluate the following integrals

$$\text{(i)} \quad \int_0^{\pi} \sin x \delta(x - \frac{\pi}{2}) dx,$$

[4 marks]

$$\text{(ii)} \quad \int_{-\infty}^{\infty} \delta(3x - 2) x^2 dx,$$

[4 marks]

$$\text{(iii)} \quad \int_0^{\infty} e^{-2x} \delta(x^2 - 5x + 6) dx$$

[5 marks]

(b) Consider the barrier function

$$f(x) = \begin{cases} b, & |x| \leq a \\ 0, & |x| > a, \end{cases}$$

with $b > 0$.

(i) Sketch $f(x)$.

[3 marks]

(ii) Calculate $\tilde{f}(k)$ the Fourier transform of $f(x)$ and sketch of the solution.

[9 marks]