UNIVERSITY OF ESWATINI

FACULTY OF SCIENCE AND ENGINEERING

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DEPARTMENT OF PHYSICS

RESIT EXAMINATION: 2019/2020

TITLE OF THE PAPER: MATHEMATICAL PHYSICS

COURSE NUMBER: PHY271

TIME ALLOWED: THREE HOURS

INSTRUCTIONS: ANSWER ANY **FOUR** OUT OF THE **FIVE** QUESTIONS. EACH QUESTION CARRIES 25 POINTS. POINTS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THE PAPER HAS 5 PAGES, INCLUDING THIS PAGE.

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Question 1

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(a) Rewrite the following complex numbers into the exponential form: $z = re^{i\theta}$.

(i) $1/(1+i)$ (ii) i^i

[6 marks]

(b) Evaluate the integral

$$\int_0^1 dx e^{ax} \cos bx,$$

for fixed real a and b.

[12 marks]

(c) Show explicitly that you can write the solution of a simple harmonic oscillator

$$Ae^{i\omega_0 t} + Be^{-i\omega_0 t} = C\cos(\omega_0 t) + D\sin(\omega_0 t) = E\cos(\omega_0 t + \phi),$$

i.e., given A and B, what are C and D, what are E and ϕ? Are there any restrictions in any of these cases?

[7 marks]

Question 2

(a) Solve the following differential equation

$$y'' + 2y' - 15y = 0, \quad y = -1, y' = 1, \text{ at } x = 0.$$

Find solution to this second order differential equation.

[9 marks]

(b) Consider the system

$$\dot{x} = -4x - y, \quad \dot{y} = x - 2y$$

(i) Determine the second order differential equations satisfied by $x(t)$.

[2 marks]

(ii) Solve the differential equation for $x(t)$ and $y(t)$. 49

[8 marks]

(iii) Find a particular solution to the system given the initial condition $x(0) = 1$ and $y(0) = 0$.

[6 marks]

Question 3

(a) The average value of a function is

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, f(t) \quad \text{or} \quad \langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} dt\, f(t)$$

as appropriate for the problem. Evaluate

(i) $\langle \sin \omega t \rangle$, (ii) $\langle \sin^2 \omega t \rangle$,

where $\omega = 2\pi/T$ and a is a constant.

[7 marks]

(b) Newton's Law of Gravitation gives the gravitational force between two masses, m and M as

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{r}.$$

(i) Prove that $\mathbf{F}$ is irrotational.

(ii) Find a scalar potential for $\mathbf{F}$.

Note that in spherical coordinates $\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$.

[8 marks]

(c) List three properties of a conservative vector field.

[6 marks]

(d) Compute the divergence of the vector field $\mathbf{v} = Ax\hat{x} + By^2\hat{y} + C\hat{z}$.

[4 marks]

Question 4

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(a) Calculate the workdone by the force $\mathbf{F} = Axy\hat{x} + B(x^2 + L^2)\hat{y}$, for moving object from point (0,0) to (L, L) along the following three different paths

(i) the line $y = 0$ from $(0, 0)$ to $(L, 0)$, followed by the line $x = L$ from $(L, 0)$ to (L, L).

(ii) the line $x = 0$ from (0,0) to (0,L), followed by the line $y = L$ from $(0, L)$ to (L, L).

(iii) the diagonal line $(0, 0)$ direct to (L, L).

[12 marks]

(b) Consider the 1D wave equation

$$\kappa\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial y^2} = 0$$

where c is a positive constant. Using the method of separation of variables, find the general solution of this equation with following boundary conditions: $u(0, t) = u(L, t) = 0$.

[13 marks]

Question 5

(a) Evaluate $\int_{-\infty}^{\infty} \delta(3x - 2)x^2 dx$

[6 marks]

(b) For the case that a function has multiple roots, $f(x_i) = 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}$$

Use this result to evaluate $\int_{-\infty}^{\infty} (x^2 - 2x + 3)\delta(x^2 - 9)dx$

[7 marks]

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(c) A damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-at}e^{i\omega_0 t}, & t \geq a \\ 0, & t < a \end{cases}$$

(i) Find $\tilde{f}(\omega)$ and

[6 marks]

(ii) the frequency distribution $|\tilde{f}(\omega)|^2$.

[3 marks]

(iii) Sketch the frequency distribution.

[3 marks]