UNIVESITY OF ESWATINI
FACULTY OF SCIENCE AND ENGINEERING
DEPARTMENT OF PHYSICS

Main Examination 2019/2020
COURSE NAME: Quantum Mechanics I
COURSE CODE: PHY341/PHY342
TIME ALLOWED: 3 hours

ANSWER ANY FIVE QUESTIONS. ALL QUESTIONS CARRY EQUAL (**20**) MARKS

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Question 1

(a). Show that if $\hat{\mathrm{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathrm{C}}$ are operators, then inj general

$$\left[\hat{\mathrm{A}}\hat{\mathrm{B}}, \hat{\mathrm{C}}\right] = \hat{\mathrm{A}}\left[\hat{\mathrm{B}}, \hat{\mathrm{C}}\right] - \left[\hat{\mathrm{A}}, \hat{\mathrm{C}}\right]\hat{\mathrm{B}}.$$

[3 marks]

(b). Given that $[\hat{x}, \hat{p}] = i\hbar$, compute $[\hat{x}, \hat{p}^n]$ and $[\hat{p}, \hat{x}^n]$.

[6 marks]

(c). If $\mathrm{V}(\hat{x})$ is a function with a convergent Taylor expansion, $\mathrm{V}(\hat{x}) = \sum_{n=0}^{\infty} c_n \hat{x}^n$, show that

$$[\hat{p}, \mathrm{V}(\hat{x})] = i\hbar \frac{d\mathrm{V}(\hat{x})}{d\hat{x}}.$$

[4 marks]

(d). For a simple harmonic oscillator, the ladder operators $\hat{a}$ and $\hat{a}^{\dagger}$ have the property that $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$. Compute

(i). $\left[\hat{a}^{\dagger}\hat{a}, \hat{a}\right]$.

[2 marks]

(ii). $\left[\hat{a}^{\dagger}\hat{a}, \hat{a}^{\dagger}\right]$.

[2 marks]

(e). The Angular momentum operator $\hat{\mathrm{L}}$ has components $\hat{\mathrm{L}}_x$, $\hat{\mathrm{L}}_y$ and $\hat{\mathrm{L}}_z$, such that $\left[\hat{\mathrm{L}}_x, \hat{\mathrm{L}}_y\right] = i\hat{\mathrm{L}}_z$, $\left[\hat{\mathrm{L}}_y, \hat{\mathrm{L}}_z\right] = i\hat{\mathrm{L}}_x$, $\left[\hat{\mathrm{L}}_z, \hat{\mathrm{L}}_x\right] = i\hat{\mathrm{L}}_y$ and $\hat{\mathrm{L}}^2 = \hat{\mathrm{L}}_x^2 + \hat{\mathrm{L}}_y^2 + \hat{\mathrm{L}}_z^2$. Compute

$$\left[\hat{\mathrm{L}}_z, \hat{\mathrm{L}}^2\right].$$

[3 marks]

Question 2

Explain what was learned about quantization of radiation from the following experiments,

(a). The photo-electric effect.

[5 marks]

(b). Franck-Hertz experiment.

[5 marks]

(c). Compton scattering.

[5 marks]

(d). The black body radiation spectrum.

[5 marks]

Question 3

Consider a Gaussian wave packet

$$\psi(x) = N \exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)$$

(a). Find the normalization constant N.

[3 marks]

(b). Find the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{x}^2 \rangle$.

[6 marks]

(c). What is the corresponding momentum space wave function $\psi(p)$?

[5 marks]

(d). Find the expectation values $\langle \hat{p} \rangle$ and $\langle \hat{p}^2 \rangle$,

[6 marks]

Question 4

A particle of mass m is confined to a one-dimensional region, $0 \leq x \leq a$, as shown in figure 1. At $t = 0$, its normalised wave function is

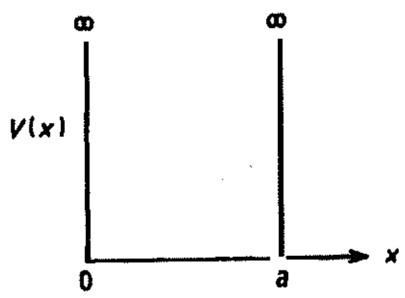


Figure 1:

$$\psi(x, t=0) = \sqrt{\frac{8}{5a}} \left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \sin\left(\frac{\pi x}{a}\right).$$

(a). What is the wave function at a later time $t = t_0$?

[10 marks]

(b). What is the average energy of the system at $t = 0$ and at $t = t_0$?

[5 marks]

(c). What is the probability that the particle is found in the left half of the box, at $t = t_0$?

[5 marks]

Question 5

(a). A quantum particle is in the state $|\psi, t\rangle$, which is a solution to the time-dependant Schrödinger equation, $i\hbar\frac{d}{dt}|\psi.t\rangle = \hat{H}|\psi, t\rangle$. Show that the expectation value of the time-independent operator $\hat{O}$ satisfy

$$\frac{d}{dt}\langle\psi, t|\hat{O}|\psi, t\rangle = \frac{1}{i\hbar}\langle\psi, t|\left[\hat{O}, \hat{H}\right]|\psi, t\rangle$$

.

[5 marks]

(b). State, with justification, whether each of the following sets of quantum numbers could be used to label a possible wave-function of the hydrogen atom.

[5 marks]

(i). $n = 1, \ell = 1, m_\ell = 1, m_s = \frac{1}{2}$
(ii). $n = 3, \ell = 1, m_\ell = 2, m_s = 0$
(iii). $n = 4, \ell = 1, m_\ell = -1, m_s = \frac{-1}{2}$
(iv). $n = 2, \ell = 1, m_\ell = 0, m_s = 0$
(v). $n = 4, \ell = 3, m_\ell = -1, m_s = \frac{1}{2}$

(c). The following wave functions are energy eigenfunctions of the hydrogen atom.

$$\Psi_1(r, \theta, \phi) = \frac{1}{\sqrt{32\pi a_0^2}}\left(2 - \frac{r}{a_0}\right)e^{\frac{-r}{2a_0}}$$

$$\Psi_2(r, \theta, \phi) = \frac{1}{81\sqrt{\pi a_0^3}}\left(\frac{r}{a_0}\right)^2 e^{\frac{-r}{3a_0}}\sin\theta\cos\theta e^{-i\phi}$$

(i). Deduce the quantum numbers n, ℓ, and m_ℓ for each wave function.

[5 marks]

(ii). Verify if $\Psi_1(r, \theta, \phi)$ is normalised and calculate the expectation value of the electron-nuclear separation in the hydrogen atom for this wave function.

[5 marks]

Question 6

(a). A infinite cubical well ("or particle in a box") has a potential, defined in Cartesian 3D coordinates as

$$V(x,y,z) = \begin{cases} 0, & \text{if } x,y,z \text{ are all between 0 and a} \\ \infty, & \text{otherwise} \end{cases}$$

(i). What is the time-dependant Schrödinger equation for the particle inside the box

[3 marks]

(ii). Use the separation of variables ansatz ($\Psi(x,y,z) = X(x)Y(y)Z(z)$) to derive **three** separate ordinary differential equations for X, Y and Z in terms of K_x, K_y and K_z, where $E = \frac{\hbar^2}{2m}(K_x^2 + K_y^2 + K_z^2)$.

[3 marks]

(iii). Find solutions to the equations above. Apply the boundary conditions to show that

$$K_x a = n_x \pi$$
$$K_y a = n_y \pi$$
$$K_z a = n_z \pi$$

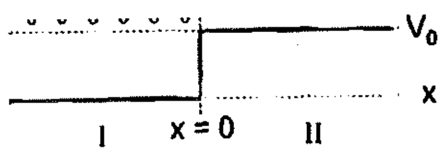
where $n_x, n_y, n_z = 1, 2, 3, ...$

[3 marks]

(b). Consider a step potential;

$$V(x) = \begin{cases} 0, & \text{if } x < 0 \\ V_0, & \text{if } x \geq 0 \end{cases}.$$

Consider the initial condition where a single particle of energy $E > V_0$ is incident from the left and no particle is incident from the right. Use



$k = \dfrac{\sqrt{2mE}}{\hbar}$ and $l = \dfrac{\sqrt{2m(E - V_0)}}{\hbar}$ to show that the time-independant Schrödinger equation for region I and II are

$$\frac{\partial^2 \Psi_I(x)}{\partial x^2} + k^2 \Psi_I(x) = 0$$
$$\frac{\partial^2 \Psi_{II}(x)}{\partial x^2} + l^2 \Psi_{II}(x) = 0.$$

[3 marks]

Appendix

1
$$\int x^2 \sin(ax)dx = \frac{2x}{a^2}\sin(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos(ax)$$

2
$$\int x^2 \cos(ax)dx = \frac{2x}{a^2}\cos(ax) + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\sin(ax)$$

3
$$\int x^2 \cos^2(bx)dx = \frac{4b^3x + 3(2b^2x^2 - 1)\sin(2bx) + 6bx\cos(2bx)}{24b^3}$$

4
$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

5 $\int_0^\infty x^n e^{\frac{-x}{a}} dx = n!a^{n+1}$ for any non-negative integer, n.

6 Planck's constant $h = 6.663 \times 10^{-34} J \cdot s$

7 Dirac's constant $\hbar = 1.05 \times 10^{-31} J \cdot s$

8 Permittivity of vaccum $\epsilon_0 = 8.854 \times 10^{-12} C^2 N^{-1} m^{-2}$

9 Ground state hydrogen energy $-E_1 = 13.6057 eV$

10 Bohr energies $E_n = \dfrac{E_1}{n^2}$

11 Hydrogen ground state wavefunction $\Psi_0 = \dfrac{1}{\sqrt{\pi a^3}} e^{\frac{-r}{a}}$

Y_0^0	$\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$
Y_1^0	$\left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$
$Y_1^{\pm1}$	$\mp\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta e^{\pm i\phi}$
Y_2^0	$\left(\frac{5}{16\pi}\right)^{\frac{1}{2}} (3\cos^2\theta - 1)$
$Y_2^{\pm1}$	$\mp\left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \sin\theta\cos\theta e^{\pm i\phi}$

Table 1: The first few spherical harmonics, $Y_l^m(\theta, \phi)$

R_{10}	$2a^{\frac{-3}{2}} e^{\frac{-r}{a}}$
R_{20}	$\frac{1}{\sqrt{2}} a^{\frac{-3}{2}} \left(1 - \frac{r}{2a}\right) e^{\frac{-r}{2a}}$
R_{21}	$\frac{1}{\sqrt{24}} a^{\frac{-3}{2}} \left(\frac{r}{a}\right) e^{\frac{-r}{2a}}$
R_{30}	$\frac{2}{\sqrt{27}} a^{\frac{-3}{2}} \left(1 - \frac{2r}{3a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right) e^{\frac{-r}{3a}}$
R_{32}	$\frac{4}{81\sqrt{30}} a^{\frac{-3}{2}} \left(\frac{r}{a}\right)^2 e^{\frac{-r}{3a}}$

Table 2: The first few hydrogen radial wavefunctions, $R_{nl}(r)$