UNIVERSITY OF ESWATINI

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2020/2021

TITLE OF PAPER: INTRODUCTION TO GENERAL RELATIVITY

COURSE NUMBER: PHY492

TIME ALLOWED: THREE HOURS

INSTRUCTIONS: ANSWER ANY FOUR OUT OF FIVE QUESTIONS.

EACH QUESTION CARRIES 25 MARKS.

MARKS FOR EACH SECTION ARE IN THE RIGHT HAND MARGIN.

THIS PAPER HAS 7 PAGES INCLUDING THE COVER PAGE.

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QUESTION 1

(a) For the following geometries write the line element and volume element:

(i) Polar coordinates (r, θ): for the Euclidean plane

$$x = r\sin\theta,$$
$$y = r\cos\theta.$$

(6 marks)

(ii) Spherical Coordinates (r, θ, ψ): for Euclidean 3-space

$$x = r\sin\theta\sin\psi,$$
$$y = r\sin\theta\cos\psi,$$
$$z = r\cos\theta.$$

(6 marks)

(iii) "Light Cone" (u, v) for $\mathbb{M}^2$

$$t = \frac{1}{2}(u+v),$$
$$x = \frac{1}{2}(v-u),$$

(6 marks)

(b) Diana leaves her twin Artemis behind on Earth and travels in her rocket for 2.2×10^8 s (~ 7 yr) of her time at $24/25 = 0.96$ the speed of light. She then instantaneously reverses her direction (fearlessly braving those gs) and returns to Earth in the same manner. Who is older at the reunion of the twins? A spacetime diagram can be very helpful.

(7 marks)

QUESTION 2

Consider a "rotating reference frame" in $\mathbb{M}^4$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

defined by the family of "observer" world lines

$$\rho = \text{constant}\,,\ z = \text{constant}\,,\ \phi = \phi_0 + \omega t$$

where t, x, y, z are Cartesian coordinates and $\rho^2 = x^2 + y^2$.

(a) Write the the spatial metric in the "rotating reference frame". HINT: introduce the coordinates (t, ρ, ϕ, z) such that

$$\begin{aligned} t &= t, \\ x &= \rho \sin\phi_0, \\ y &= \rho\cos\phi_0, \\ \rho &= x^2 + y^2, \\ \phi &= \phi_0 + \omega t, \\ z &= z, \end{aligned}$$

(5 marks)

(b) Give the circumference of a (spatial) circle in the equatorial plane, centered at the origin and the ratio of circumference to radius. Compare this with In Fitzgerald-Lorentz contraction

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

(7 marks)

(c) Consider two simultaneous clocks P and Q on a circle with $\rho = constant$ and $z = constant$, can these clocks be synchronized around such a circle?

(5 marks)

(d) Given that the radius of the earth at the equator is $\rho_{eq} = 6\,378.1370km$ and the angular velocity of the earth is $\omega = 7.27 \times 10^{-5} radians/s$, $\rho\omega = 0.465.1km/s$. What is the ratio of the earth's circumference to radius and the difference between clocks in part (c)?

(5 marks)

(e) Can such a reference frame exist?

(3 marks)

QUESTION 3

Consider a metric in a 2+1 dimension space:

$$g_{\mu\nu} = \begin{pmatrix} -e^{-\alpha x} & 0 & 0 \\ 0 & e^{-\alpha x} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where α is a positive, non-zero constant, and the coordinates are (t, x, y).

(a) Compute all non-zero Christoffel symbols.

(5 marks)

(b) Compute all non-zero Riemann tensor terms. From that, please determine an approximate analytic relationship for the density as a function of position. HINT: At most, the density will be a function of the x coordinate only.

(5 marks)

(c) A particle starts at rest at the origin. Compute the instantaneous 4-acceleration on the particle, $dU^\mu/d\tau$.

(5 marks)

(d) The system is left to evolve, but at some later time, the particle is found at position, x_f (the 1st spatial component, just to be clear). Using whatever conservation laws you like, please compute U^1 at that time. (For small displacements, compute the velocity component to lowest order in x.

(5 marks)

(e) What is the volume element of this metric?

(5 marks)

QUESTION 4

For homogeneous, isotropic cosmology

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2)$$

(a) Calculate the non-vanishing components of the Christoffel symbol.

(8 marks)

(b) Calculate the independent Ricci tensor components and the Ricci scalar.

(7 marks)

(c) Show that the general evolution equations for homogeneous, isotropic cosmology with a cosmological constant is

$$3\frac{\dot{a}^2}{a^2} = 8\pi\rho + \Lambda$$
$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3P) + \Lambda$$

(10 marks)

QUESTION 5

Consider Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dx^2 + r^2 d\Omega^2$$

(a) When does the metric components become singular?

(6 marks)

(b) What is an event horizon? What is the event horizon of Schwarzschild metric?

(6 marks)

(c) Who will age slower (who will have aged less after a fixed coordinate time): an observer orbiting a black hole at radius r, or an observer dangling above a black hole at radius, r?

(6 marks)

(d) In the limit of $r \gg 2M$, what is the approximate ratio of the distortion effects between the two observers?

(7 marks)

Appendix

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\begin{aligned}
\Gamma^{\gamma}_{\alpha\beta} &= \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta} \right) \\
R^{\delta}_{\gamma\alpha\beta} &= \partial_\alpha \Gamma^{\delta}_{\beta\gamma} - \partial_\beta \Gamma^{\delta}_{\alpha\gamma} + \Gamma^{\delta}_{\alpha\eta}\Gamma^{\eta}_{\beta\gamma} - \Gamma^{\delta}_{\beta\eta}\Gamma^{\eta}_{\alpha\gamma} \\
R_{ab} &= R^{d}_{abd} = \partial_b \Gamma^{d}_{ad} - \partial_d \Gamma^{d}_{ab} + \Gamma^{e}_{ad}\Gamma^{d}_{eb} - \Gamma^{e}_{ab}\Gamma^{d}_{ed} \\
R &= g^{ab} R_{ab} \\
G_{ab} &= 8\pi T_{ab} \\
G_{ab} &= R_{ab} - \frac{1}{2} g_{ab} R \\
T_{ab} &= \rho u_a u_b + P(g_{ab} + u_a u_b)
\end{aligned}
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