UNIVERSITY OF SWAZILAND

FINAL EXAMINATION, 2016/2017

B.Sc. II, B.Eng II, BASS II, BED. II

Title of Paper : Linear Algebra

Course Number : MAT221/M220

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:
   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.
   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**
2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

A1. (a) Let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of a vector space V. Explain precisely what is meant by each of the following statements

(i) S spans V (2)

(ii) S is linearly dependent in V (2)

(iii) S is a basis for V. (2)

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - z \\ x + y - z \\ z \end{pmatrix}$$

(i) Find the matrix A of T with respect to the standard basis.

(ii) Find the matrix A' of T with respect to the basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(iii) Find a 3×3 transition matrix P

(iv) Show that $A' = PAP^{-1}$

A2. (a) Let V be a vector space, A and B be finite sets of non-zero vectors in V such that $A \subset B$ show that

(i) A linearly dependent $\Rightarrow$ B is also linearly dependent

(ii) B is linearly independent $\Rightarrow$ A is also linearly independent (10)

(b) (i) Express A and A^{-1} as a product of elementary matrices where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

(ii) Compute the product for A^{-1} and show that A^{-1} is the inverse of A. (6)

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

(a) Find the characteristic polynomial, eigenvectors and eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

(14)

(b) Use Cramer's rule to solve

$$\begin{pmatrix} 2 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(6)

QUESTION B4 [*20 Marks*]

(a) Find values of k for which the linear system has

(i) a unique solution

(ii) no solution

(iii) infinitely many solutions

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & k^2-5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}$$

(10)

(b) Find the coordinate vector of $\begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix}$ with respect to

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(6)

(c) Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ where $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ x \\ 4x-2y \end{pmatrix}$

(4)

QUESTION B5 [*20 Marks*]

(a) Let V be the set of all ordered pairs of real numbers. Define addition and scalar multiplication as follows

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$$

$$\alpha(x, y) = (\alpha x_1 + \alpha - 1, \alpha y_1 + \alpha - 1)$$

Show that V is a vector space. (10)

(b) Show that the vector $(4, 2, -6)$ is a linear combination of the vectors $(4, 2, -3)$, $(2, 1, -2)$ and $(-2, -1, 0)$ (6)

(c) Determine whether the following has a non-trivial solution

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

(4)

QUESTION B6 [*20 Marks*]

(a) Determine whether the following mappings are linear transformations

(i) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ 2x + y \end{pmatrix}$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 1 - z \\ x + y \\ x + 2y \end{pmatrix}$ (10)

(b) Prove that the set $B = \{x^2 + 1, x - 1, 2x + 2\}$ is a basis for the vector space $V = P_2(x)$ (10)

QUESTION B7 [*20 Marks*]

(a) Let $s = \{v_2, v_2, \cdots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that s is linearly dependent if and only if one of the vectors v_j is a linear combination of the preceeding vectors in s. (10)

(b) Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

(10)

END OF EXAMINATION PAPER