UNIVERSITY OF SWAZILAND

MAIN EXAMINATION PAPER 2008

TITLE OF PAPER : PROBABILITY THEORY

COURSE CODE : ST 201

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS : ANSWER ANY FOUR QUESTIONS.

REQUIREMENTS: SCIENTIFIC CALCULATOR.

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QUESTION ONE

ATM card holders with a certain bank are each assigned a 4-digit PIN (personal identity number). How many different possible PINs are there under each of the following conditions?

a) The first digit may not be zero.

(3 Marks)

b) The first digit may not be zero, and the four digits may not all be the same.

(3 Marks)

c) No zeros are allowed in any position and all four digits must be different.

(4 Marks)

d) Zeros are allowed in all positions but sequences of four consecutive digits up (e.g. 2 3 4 5) or down (e.g. 3 2 1 0) are not allowed.

(4 Marks)

e) Zeros are allowed in all positions but no digit may occur more than twice.

(6 Marks)

QUESTION TWO

The random variable X has probability density function f(x) given by

$$f(x) = kx^2(1-x), \qquad 0 \le x \le 1$$

a) Show that $k = 12$.

(2 Marks)

b) Show that the mode of X is at $x = 2/3$ and draw a graph of $f(x)$.

(5 Marks)

c) Find the mean and variance of X.

(5 Marks)

d) Find the cumulative distribution function of X and obtain the probability that X lies within one standard deviation of its mean.

(8 Marks)

QUESTION THREE

The total claim amount X made in one year on a portfolio of insurance policies has probability density function;

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & x \geq 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that

i) $E(X) = 2/\lambda$

ii) $Var(X) = 2/\lambda^2$

iii) $P(X > x) = e^{-\lambda x}(1 + \lambda x)$

(9 Marks)

b) If X is measured in units of E1000, λ may be assumed to take the value 0.01. The company has a total sum (policyholders' premiums + reserves) of E500,000 available to meet the year's claims. Show that the probability that the company is ruined (i.e. $P(X > 500)$) is 0.040 (to 3 decimal places).

(4 Marks)

c) A trainee actuary mistakenly assumes the distribution of total claim amount to be Normal with the same mean and variance as X (taking $\lambda = 0.01$). On this assumption, find the probability of ruin, given that E500,000 is available to meet the year's claims.

(4 Marks)

d) Making this mistaken assumption of Normality, the trainee calculates that E450,000 is the sum to be set aside to meet the year's claims with a probability of ruin of less than 0.04. What is the true probability of ruin, if only E450,000 is available to meet the year's claims?

(3 Marks)

QUESTION FOUR

a) The random variables X_1, ..., X_4 have a joint p.d.f. given by

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2) & 0 < x_i < 1, i = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

i) Use the joint probability density function to compute $P(X_1 < 1/2, X_2 < 3/4, X_3 < 1, X_4 > 1/2)$

(8 Marks)

ii) Compute the marginal probability density function of (X_1, X_2)

(7 Marks)

b) If X and Y have the joint p.d.f. $f(x, y) = ce^{-x} e^{-2y}$ for $0 < x < 1$ and $x < y < 1$, what the value of the constant is c?

(5 Marks)

QUESTION FIVE

a) Let X have a uniform(0,1) distribution and Z have a uniform(0,0.1) distribution. Suppose X and Z are independent. Let Y = X + Z and consider the random vector (X, Y). The joint pdf of (X, Y) is

$$f(x, y) = 10, \quad 0 < x < 1, \quad x < y < x + 0.1$$

Show that $\rho_{xy} = (100/101)^{1/2}$.

(10 Marks)

b) In a group of 60 people, the numbers who do or do not smoke and do or do not have cancer are reported as shown in table below. Let be the sample space consisting of these 60 people. A person is chosen at random from the group.

	Not smoke	Smoke	Total
Not cancer	40	10	50
Cancer	7	3	10
Totals	47	13	60

i) Show that smoking and cancer status are independent.

(3 Marks)

ii) Calculate $P(C=1)$ and $P(C=1 | S=1)$

(7 Marks)

QUESTION SIX

Let the continuous random vector (X, Y) have joint pdf

$$f(x, y) = e^{-y}, \quad 0 < x < y < \infty$$

Derive the

i) marginal distributionof X,

(4 Marks)

ii) conditional distribution of Y given X,

(6 Marks)

iii) conditional mean and variance of Y given X.

(10 Marks)