UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2009

TITLE OF PAPER	**: SAMPLE SURVEY THEORY**
COURSE CODE	**: ST 306**
TIME ALLOWED	**: TWO (2) HOURS**
REQUIREMENTS	**: CALCULATOR AND STATISTICAL TABLES** **FORMULA SHEET ATTACHED**
INSTRUCTIONS	**: ANSWER ANY THREE QUESTIONS**

Question 1

The wholesale price paid for oranges in large shipments is based on the sugar content of the load. The exact sugar content cannot be determined prior to the purchase and extraction of the juice from the entire load. You may assume that (a) the sugar content of an individual orange, y is closely related to its weight, x; and (b) the ratio of the total sugar content τ_y to the total weight of the truckload τ_x is equal to the ratio of the mean sugar content per orange, μ_y to the mean weight, μ_x.

a) How can the total sugar content of the load be estimated from a random sample *n* oranges from the load
   i) if the total number of oranges, N in the load is known?
   ii) if only the total weight of the oranges, τ_x in the truck is known?

   In each case, what measurements must be made on the sample of n oranges?

   (5 Marks)

b) Variates y_i and x_i are measured on each unit of a simple random sample of size n, assumed large. Show that the variance of $r = \bar{y}/\bar{x}$ is

$$V(r) \approx \frac{1-f}{n\mu_x^2} \sum_{i=1}^{N} \frac{(y_i - Rx_i)^2}{N-1}$$

   where $R = \mu_y / \mu_x$ is the ratio of the population means and $f = n/N$.

   (5 Marks)

c) Hence derive an approximate expression for the variance of your estimator in (ii) of part (a) above. State the condition under which the use of your estimator is better than the use of the estimator in (i) of part (a).

   (5 Marks)

d) Roughly how many oranges must be sampled from a very large truckload of oranges weighing 818 Kgs in order for the standard error of the estimator in (ii) of part (a) to be about 1.36 Kgs, where

$$\sum_{i=1}^{N} \frac{(y_i - Rx_i)^2}{N-1} = (0.0030)^2$$

   You may assume that the mean weight of an orange is 0.18 Kgs.

   (5 Marks)

Question 2

(a) Define the term *stratified random sampling*. Explain what is meant by the expressions *stratification with proportional allocation* and *stratification with optimal allocation.*

(b) A population of size N is divided into L strata, the stratum h being of size N_h. The mean of the elements in the h^{th} stratum is $\bar{Y}_h$ and the variance is S_h^2. The population mean is $\bar{Y}$ and the variance is S^2. A sample of size n is selected by taking independent random samples of size n_h from stratum h.

The population mean $\bar{Y}$ is to be estimated, either using a stratified random sample with proportional allocation or using a simple random sample of the same size. Let V_{prop} and V_{ran} be the variances of the two estimators. Show that

$$V_{ran} = V_{prop} + \frac{N-n}{nN(N-1)} \left[\sum^{L} N_h(\bar{Y}_h - \bar{Y})^2 - \frac{1}{N}\sum^{L}(N-N_h)S_h^2\right]$$

(c) The following data show the stratification of all farms in a country by farm size showing summary information about the area devoted to corn (maize) per farm in each stratum.

Farm Size (acres)	*Number of Farms* N_h	*Average Corn Acres* $\bar{Y}_h$	*Standard Deviation* S_h
≤ 40	394	5.4	8.3
41- 80	461	16.3	13.3
81- 120	391	24.3	15.1
121- 160	334	34.5	19.8
161- 200	169	42.1	24.5
201- 240	113	50.1	26.0
≥ 241	148	63.8	35.2
Overall	2010	26.3	

$\sum^{L} W_h S_h^2 = 343.2788$; $\sum^{L} W_h S_h = 17.0183$, where $W_h = \frac{N_h}{N}$ for $h = 1,2, \ldots, L$

For a sample of 100 farms, compute the sample sizes in each stratum under:

(i) Optimum allocation

(ii) Proportional allocation.

Compare the precisions of these methods with that of simple random sampling.

(20 Marks)

Question 3

At an experimental station 100 fields (each of area one hectare) were sown with wheat. Each field was divided into sixteen equal plots. A simple random sample of 10 fields was selected, and from each of these fields a simple random sample of 4 plots was selected. The yields in kg/plot are given below:

	Plot within field			
Field	*1*	*2*	*3*	*4*
1	4.28	4.36	3.00	3.52
2	4.20	4.66	3.64	5.00
3	4.40	4.72	4.04	3.98
4	5.16	4.24	4.96	3.84
5	4.08	3.96	3.42	3.08
6	4.12	4.68	3.46	4.02
7	4.00	4.84	4.32	3.72
8	3.06	4.24	4.76	3.12
9	4.16	4.36	3.50	5.00
10	4.32	4.84	3.96	4.04

a) Estimate the wheat yield per hectare for the experimental station and its standard error.

(8 Marks)

b) Estimate the relative efficiency of this estimator to that obtained from a simple random sample of 40 plots.

(4 Marks)

c) If the cost of including a field in the sample is four times the cost of including an extra plot, and total cost (excluding overheads) must not exceed 100 units, use the method of Lagrange multipliers to derive the optimum number of fields and the optimum number of plots per field for the sample.

(8 Marks)

Question 4

A simple random sample of 10 hospitals was selected from a population of 33 hospitals that had received state funding to upgrade their emergency medical services. Within each of the selected hospitals, the records of all patients hospitalised in the past 12 months for traumatic injuries (i.e. accidents, poisonings, violence, burns, etc) were examined. The numbers of patients hospitalised for trauma conditions and the numbers discharged dead for the selected hospitals are given below.

Hospital	*Number of patients hospitalised for trauma conditions*	*Number with trauma conditions discharged dead*
1	560	4
2	190	4
3	260	2
4	370	4
5	190	4
6	130	0
7	170	9
8	170	2
9	60	0
10	110	1

a) Explain why this design may be considered as a cluster sample. What are the first-stage and second-stage units?

(2 Marks)

b)

i) Obtain a point estimate and an approximate 95% confidence interval for the total number of persons hospitalised for trauma conditions for the 33 hospitals. State the properties of your estimator.

ii) Obtain a point estimate of the proportion of persons discharged dead among those hospitalised for trauma conditions for the 33 hospitals, using the cluster totals. Hence calculate an approximate 95% confidence interval for this proportion, and comment on the validity of the assumptions necessary for this calculation.

(12 marks)

c) Give reasons why, for this survey, cluster sampling might be preferred to stratified random sampling. What might be the drawbacks of cluster sampling? Discuss, with reasons, any improvements you might make if another survey was being planned on the same topic.

(6 Marks)

STATISTICAL TABLES

Cumulative normal distribution

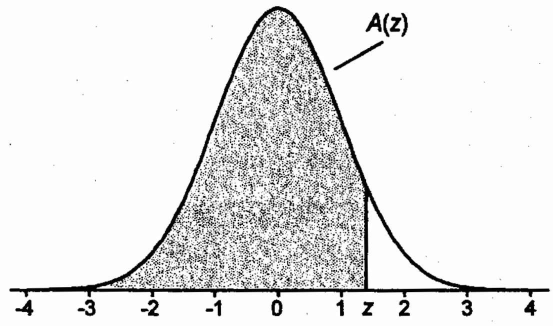
Critical values of the *t* distribution

Critical values of the *F* distribution

Critical values of the chi-squared distribution

TABLE A.1

Cumulative Standardized Normal Distribution



$A(z)$ is the integral of the standardized normal distribution from $-\infty$ to z (in other words, the area under the curve to the left of z). It gives the probability of a normal random variable not being more than z standard deviations above its mean. Values of z of particular importance:

z	$A(z)$	
1.645	0.9500	Lower limit of right 5% tail
1.960	0.9750	Lower limit of right 2.5% tail
2.326	0.9900	Lower limit of right 1% tail
2.576	0.9950	Lower limit of right 0.5% tail
3.090	0.9990	Lower limit of right 0.1% tail
3.291	0.9995	Lower limit of right 0.05% tail

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999							

TABLE A.2

t Distribution: Critical Values of *t*

Degrees of freedom	Significance level — Two-tailed test: / One-tailed test:	10% / 5%	5% / 2.5%	2% / 1%	1% / 0.5%	0.2% / 0.1%	0.1% / 0.05%
1		6.314	12.706	31.821	63.657	318.309	636.619
2		2.920	4.303	6.965	9.925	22.327	31.599
3		2.353	3.182	4.541	5.841	10.215	12.924
4		2.132	2.776	3.747	4.604	7.173	8.610
5		2.015	2.571	3.365	4.032	5.893	6.869
6		1.943	2.447	3.143	3.707	5.208	5.959
7		1.894	2.365	2.998	3.499	4.785	5.408
8		1.860	2.306	2.896	3.355	4.501	5.041
9		1.833	2.262	2.821	3.250	4.297	4.781
10		1.812	2.228	2.764	3.169	4.144	4.587
11		1.796	2.201	2.718	3.106	4.025	4.437
12		1.782	2.179	2.681	3.055	3.930	4.318
13		1.771	2.160	2.650	3.012	3.852	4.221
14		1.761	2.145	2.624	2.977	3.787	4.140
15		1.753	2.131	2.602	2.947	3.733	4.073
16		1.746	2.120	2.583	2.921	3.686	4.015
17		1.740	2.110	2.567	2.898	3.646	3.965
18		1.734	2.101	2.552	2.878	3.610	3.922
19		1.729	2.093	2.539	2.861	3.579	3.883
20		1.725	2.086	2.528	2.845	3.552	3.850
21		1.721	2.080	2.518	2.831	3.527	3.819
22		1.717	2.074	2.508	2.819	3.505	3.792
23		1.714	2.069	2.500	2.807	3.485	3.768
24		1.711	2.064	2.492	2.797	3.467	3.745
25		1.708	2.060	2.485	2.787	3.450	3.725
26		1.706	2.056	2.479	2.779	3.435	3.707
27		1.703	2.052	2.473	2.771	3.421	3.690
28		1.701	2.048	2.467	2.763	3.408	3.674
29		1.699	2.045	2.462	2.756	3.396	3.659
30		1.697	2.042	2.457	2.750	3.385	3.646
32		1.694	2.037	2.449	2.738	3.365	3.622
34		1.691	2.032	2.441	2.728	3.348	3.601
36		1.688	2.028	2.434	2.719	3.333	3.582
38		1.686	2.024	2.429	2.712	3.319	3.566
40		1.684	2.021	2.423	2.704	3.307	3.551
42		1.682	2.018	2.418	2.698	3.296	3.538
44		1.680	2.015	2.414	2.692	3.286	3.526
46		1.679	2.013	2.410	2.687	3.277	3.515
48		1.677	2.011	2.407	2.682	3.269	3.505
50		1.676	2.009	2.403	2.678	3.261	3.496
60		1.671	2.000	2.390	2.660	3.232	3.460
70		1.667	1.994	2.381	2.648	3.211	3.435
80		1.664	1.990	2.374	2.639	3.195	3.416
90		1.662	1.987	2.368	2.632	3.183	3.402
100		1.660	1.984	2.364	2.626	3.174	3.390
120		1.658	1.980	2.358	2.617	3.160	3.373
150		1.655	1.976	2.351	2.609	3.145	3.357
200		1.653	1.972	2.345	2.601	3.131	3.340
300		1.650	1.968	2.339	2.592	3.118	3.323
400		1.649	1.966	2.336	2.588	3.111	3.315
500		1.648	1.965	2.334	2.586	3.107	3.310
600		1.647	1.964	2.333	2.584	3.104	3.307
∞		1.645	1.960	2.326	2.576	3.090	3.291

Table A.3

F Distribution: Critical Values of *F* (5% significance level)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.36	246.46	247.32	248.01
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.42	19.43	19.44	19.45
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.71	8.69	8.67	8.66
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.87	5.84	5.82	5.80
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.64	4.60	4.58	4.56
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.96	3.92	3.90	3.87
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.53	3.49	3.47	3.44
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.24	3.20	3.17	3.15
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.03	2.99	2.96	2.94
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.86	2.83	2.80	2.77
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.74	2.70	2.67	2.65
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.64	2.60	2.57	2.54
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.55	2.51	2.48	2.46
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.48	2.44	2.41	2.39
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.42	2.38	2.35	2.33
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.37	2.33	2.30	2.28
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.33	2.29	2.26	2.23
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.29	2.25	2.22	2.19
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.26	2.21	2.18	2.16
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.22	2.18	2.15	2.12
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.20	2.16	2.12	2.10
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.17	2.13	2.10	2.07
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.15	2.11	2.08	2.05
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.13	2.09	2.05	2.03
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.11	2.07	2.04	2.01
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.09	2.05	2.02	1.99
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.08	2.04	2.00	1.97
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.06	2.02	1.99	1.96
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.05	2.01	1.97	1.94
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.04	1.99	1.96	1.93
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.04	1.99	1.94	1.91	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.95	1.90	1.87	1.84
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.89	1.85	1.81	1.78
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.86	1.82	1.78	1.75
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.89	1.84	1.79	1.75	1.72
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.82	1.77	1.73	1.70
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.86	1.80	1.76	1.72	1.69
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.79	1.75	1.71	1.68
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.78	1.73	1.69	1.66
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.82	1.76	1.71	1.67	1.64
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.80	1.74	1.69	1.66	1.62
250	3.88	3.03	2.64	2.41	2.25	2.13	2.05	1.98	1.92	1.87	1.79	1.73	1.68	1.65	1.61
300	3.87	3.03	2.63	2.40	2.24	2.13	2.04	1.97	1.91	1.86	1.78	1.72	1.68	1.64	1.61
400	3.86	3.02	2.63	2.39	2.24	2.12	2.03	1.96	1.90	1.85	1.78	1.72	1.67	1.63	1.60
500	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.77	1.71	1.66	1.62	1.59
600	3.86	3.01	2.62	2.39	2.23	2.11	2.02	1.95	1.90	1.85	1.77	1.71	1.66	1.62	1.59
750	3.85	3.01	2.62	2.38	2.23	2.11	2.02	1.95	1.89	1.84	1.77	1.70	1.66	1.62	1.58
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.76	1.70	1.65	1.61	1.58

Table A.3 (continued)

F Distribution: Critical Values of F (5% significance level)

ν_2 \ ν_1	25	30	35	40	50	60	75	100	150	200
1	249.26	250.10	250.69	251.14	251.77	252.20	252.62	253.04	253.46	253.68
2	19.46	19.46	19.47	19.47	19.48	19.48	19.48	19.49	19.49	19.49
3	8.63	8.62	8.60	8.59	8.58	8.57	8.56	8.55	8.54	8.54
4	5.77	5.75	5.73	5.72	5.70	5.69	5.68	5.66	5.65	5.65
5	4.52	4.50	4.48	4.46	4.44	4.43	4.42	4.41	4.39	4.39
6	3.83	3.81	3.79	3.77	3.75	3.74	3.73	3.71	3.70	3.69
7	3.40	3.38	3.36	3.34	3.32	3.30	3.29	3.27	3.26	3.25
8	3.11	3.08	3.06	3.04	3.02	3.01	2.99	2.97	2.96	2.95
9	2.89	2.86	2.84	2.83	2.80	2.79	2.77	2.76	2.74	2.73
10	2.73	2.70	2.68	2.66	2.64	2.62	2.60	2.59	2.57	2.56
11	2.60	2.57	2.55	2.53	2.51	2.49	2.47	2.46	2.44	2.43
12	2.50	2.47	2.44	2.43	2.40	2.38	2.37	2.35	2.33	2.32
13	2.41	2.38	2.36	2.34	2.31	2.30	2.28	2.26	2.24	2.23
14	2.34	2.31	2.28	2.27	2.24	2.22	2.21	2.19	2.17	2.16
15	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.10	2.10
16	2.23	2.19	2.17	2.15	2.12	2.11	2.09	2.07	2.05	2.04
17	2.18	2.15	2.12	2.10	2.08	2.06	2.04	2.02	2.00	1.99
18	2.14	2.11	2.08	2.06	2.04	2.02	2.00	1.98	1.96	1.95
19	2.11	2.07	2.05	2.03	2.00	1.98	1.96	1.94	1.92	1.91
20	2.07	2.04	2.01	1.99	1.97	1.95	1.93	1.91	1.89	1.88
21	2.05	2.01	1.98	1.96	1.94	1.92	1.90	1.88	1.86	1.84
22	2.02	1.98	1.96	1.94	1.91	1.89	1.87	1.85	1.83	1.82
23	2.00	1.96	1.93	1.91	1.88	1.86	1.84	1.82	1.80	1.79
24	1.97	1.94	1.91	1.89	1.86	1.84	1.82	1.80	1.78	1.77
25	1.96	1.92	1.89	1.87	1.84	1.82	1.80	1.78	1.76	1.75
26	1.94	1.90	1.87	1.85	1.82	1.80	1.78	1.76	1.74	1.73
27	1.92	1.88	1.86	1.84	1.81	1.79	1.76	1.74	1.72	1.71
28	1.91	1.87	1.84	1.82	1.79	1.77	1.75	1.73	1.70	1.69
29	1.89	1.85	1.83	1.81	1.77	1.75	1.73	1.71	1.69	1.67
30	1.88	1.84	1.81	1.79	1.76	1.74	1.72	1.70	1.67	1.66
35	1.82	1.79	1.76	1.74	1.70	1.68	1.66	1.63	1.61	1.60
40	1.78	1.74	1.72	1.69	1.66	1.64	1.61	1.59	1.56	1.55
50	1.73	1.69	1.66	1.63	1.60	1.58	1.55	1.52	1.50	1.48
60	1.69	1.65	1.62	1.59	1.56	1.53	1.51	1.48	1.45	1.44
70	1.66	1.62	1.59	1.57	1.53	1.50	1.48	1.45	1.42	1.40
80	1.64	1.60	1.57	1.54	1.51	1.48	1.45	1.43	1.39	1.38
90	1.63	1.59	1.55	1.53	1.49	1.46	1.44	1.41	1.38	1.36
100	1.62	1.57	1.54	1.52	1.48	1.45	1.42	1.39	1.36	1.34
120	1.60	1.55	1.52	1.50	1.46	1.43	1.40	1.37	1.33	1.32
150	1.58	1.54	1.50	1.48	1.44	1.41	1.38	1.34	1.31	1.29
200	1.56	1.52	1.48	1.46	1.41	1.39	1.35	1.32	1.28	1.26
250	1.55	1.50	1.47	1.44	1.40	1.37	1.34	1.31	1.27	1.25
300	1.54	1.50	1.46	1.43	1.39	1.36	1.33	1.30	1.26	1.23
400	1.53	1.49	1.45	1.42	1.38	1.35	1.32	1.28	1.24	1.22
500	1.53	1.48	1.45	1.42	1.38	1.35	1.31	1.28	1.23	1.21
600	1.52	1.48	1.44	1.41	1.37	1.34	1.31	1.27	1.23	1.20
750	1.52	1.47	1.44	1.41	1.37	1.34	1.30	1.26	1.22	1.20
1000	1.52	1.47	1.43	1.41	1.36	1.33	1.30	1.26	1.22	1.19

TABLE A.3 (continued)

F Distribution: Critical Values of *F* (1% significance level)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47	6055.85	6106.32	6142.67	6170.10	6191.53	6208.73
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.44	99.44	99.45
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.92	26.83	26.75	26.69
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.25	14.15	14.08	14.02
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.77	9.68	9.61	9.55
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.60	7.52	7.45	7.40
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.36	6.28	6.21	6.16
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.56	5.48	5.41	5.36
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	5.01	4.92	4.86	4.81
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.60	4.52	4.46	4.41
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.29	4.21	4.15	4.10
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.05	3.97	3.91	3.86
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.86	3.78	3.72	3.66
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.70	3.62	3.56	3.51
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.56	3.49	3.42	3.37
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.45	3.37	3.31	3.26
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.35	3.27	3.21	3.16
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.27	3.19	3.13	3.08
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.19	3.12	3.05	3.00
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.13	3.05	2.99	2.94
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.07	2.99	2.93	2.88
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	3.02	2.94	2.88	2.83
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.97	2.89	2.83	2.78
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.93	2.85	2.79	2.74
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.89	2.81	2.75	2.70
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.86	2.78	2.72	2.66
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.82	2.75	2.68	2.63
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.79	2.72	2.65	2.60
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.77	2.69	2.63	2.57
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.74	2.66	2.60	2.55
35	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88	2.74	2.64	2.56	2.50	2.44
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.56	2.48	2.42	2.37
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.46	2.38	2.32	2.27
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.39	2.31	2.25	2.20
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.45	2.35	2.27	2.20	2.15
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.42	2.31	2.23	2.17	2.12
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61	2.52	2.39	2.29	2.21	2.14	2.09
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.37	2.27	2.19	2.12	2.07
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.23	2.15	2.09	2.03
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.31	2.20	2.12	2.06	2.00
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.27	2.17	2.09	2.03	1.97
250	6.74	4.69	3.86	3.40	3.09	2.87	2.71	2.58	2.48	2.39	2.26	2.15	2.07	2.01	1.95
300	6.72	4.68	3.85	3.38	3.08	2.86	2.70	2.57	2.47	2.38	2.24	2.14	2.06	1.99	1.94
400	6.70	4.66	3.83	3.37	3.06	2.85	2.68	2.56	2.45	2.37	2.23	2.13	2.05	1.98	1.92
500	6.69	4.65	3.82	3.36	3.05	2.84	2.68	2.55	2.44	2.36	2.22	2.12	2.04	1.97	1.92
600	6.68	4.64	3.81	3.35	3.05	2.83	2.67	2.54	2.44	2.35	2.21	2.11	2.03	1.96	1.91
750	6.67	4.63	3.81	3.34	3.04	2.83	2.66	2.53	2.43	2.34	2.21	2.11	2.02	1.96	1.90
1000	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43	2.34	2.20	2.10	2.02	1.95	1.90

Table A.3 (continued)

F Distribution: Critical Values of *F* (1% significance level)

ν_2 \ ν_1	25	30	35	40	50	60	75	100	150	200
1	6239.83	6260.65	6275.57	6286.78	6302.52	6313.03	6323.56	6334.11	6344.68	6349.97
2	99.46	99.47	99.47	99.47	99.48	99.48	99.49	99.49	99.49	99.49
3	26.58	26.50	26.45	26.41	26.35	26.32	26.28	26.24	26.20	26.18
4	13.91	13.84	13.79	13.75	13.69	13.65	13.61	13.58	13.54	13.52
5	9.45	9.38	9.33	9.29	9.24	9.20	9.17	9.13	9.09	9.08
6	7.30	7.23	7.18	7.14	7.09	7.06	7.02	6.99	6.95	6.93
7	6.06	5.99	5.94	5.91	5.86	5.82	5.79	5.75	5.72	5.70
8	5.26	5.20	5.15	5.12	5.07	5.03	5.00	4.96	4.93	4.91
9	4.71	4.65	4.60	4.57	4.52	4.48	4.45	4.41	4.38	4.36
10	4.31	4.25	4.20	4.17	4.12	4.08	4.05	4.01	3.98	3.96
11	4.01	3.94	3.89	3.86	3.81	3.78	3.74	3.71	3.67	3.66
12	3.76	3.70	3.65	3.62	3.57	3.54	3.50	3.47	3.43	3.41
13	3.57	3.51	3.46	3.43	3.38	3.34	3.31	3.27	3.24	3.22
14	3.41	3.35	3.30	3.27	3.22	3.18	3.15	3.11	3.08	3.06
15	3.28	3.21	3.17	3.13	3.08	3.05	3.01	2.98	2.94	2.92
16	3.16	3.10	3.05	3.02	2.97	2.93	2.90	2.86	2.83	2.81
17	3.07	3.00	2.96	2.92	2.87	2.83	2.80	2.76	2.73	2.71
18	2.98	2.92	2.87	2.84	2.78	2.75	2.71	2.68	2.64	2.62
19	2.91	2.84	2.80	2.76	2.71	2.67	2.64	2.60	2.57	2.55
20	2.84	2.78	2.73	2.69	2.64	2.61	2.57	2.54	2.50	2.48
21	2.79	2.72	2.67	2.64	2.58	2.55	2.51	2.48	2.44	2.42
22	2.73	2.67	2.62	2.58	2.53	2.50	2.46	2.42	2.38	2.36
23	2.69	2.62	2.57	2.54	2.48	2.45	2.41	2.37	2.34	2.32
24	2.64	2.58	2.53	2.49	2.44	2.40	2.37	2.33	2.29	2.27
25	2.60	2.54	2.49	2.45	2.40	2.36	2.33	2.29	2.25	2.23
26	2.57	2.50	2.45	2.42	2.36	2.33	2.29	2.25	2.21	2.19
27	2.54	2.47	2.42	2.38	2.33	2.29	2.26	2.22	2.18	2.16
28	2.51	2.44	2.39	2.35	2.30	2.26	2.23	2.19	2.15	2.13
29	2.48	2.41	2.36	2.33	2.27	2.23	2.20	2.16	2.12	2.10
30	2.45	2.39	2.34	2.30	2.25	2.21	2.17	2.13	2.09	2.07
35	2.35	2.28	2.23	2.19	2.14	2.10	2.06	2.02	1.98	1.96
40	2.27	2.20	2.15	2.11	2.06	2.02	1.98	1.94	1.90	1.87
50	2.17	2.10	2.05	2.01	1.95	1.91	1.87	1.82	1.78	1.76
60	2.10	2.03	1.98	1.94	1.88	1.84	1.79	1.75	1.70	1.68
70	2.05	1.98	1.93	1.89	1.83	1.78	1.74	1.70	1.65	1.62
80	2.01	1.94	1.89	1.85	1.79	1.75	1.70	1.65	1.61	1.58
90	1.99	1.92	1.86	1.82	1.76	1.72	1.67	1.62	1.57	1.55
100	1.97	1.89	1.84	1.80	1.74	1.69	1.65	1.60	1.55	1.52
120	1.93	1.86	1.81	1.76	1.70	1.66	1.61	1.56	1.51	1.48
150	1.90	1.83	1.77	1.73	1.66	1.62	1.57	1.52	1.46	1.43
200	1.87	1.79	1.74	1.69	1.63	1.58	1.53	1.48	1.42	1.39
250	1.85	1.77	1.72	1.67	1.61	1.56	1.51	1.46	1.40	1.36
300	1.84	1.76	1.70	1.66	1.59	1.55	1.50	1.44	1.38	1.35
400	1.82	1.75	1.69	1.64	1.58	1.53	1.48	1.42	1.36	1.32
500	1.81	1.74	1.68	1.63	1.57	1.52	1.47	1.41	1.34	1.31
600	1.80	1.73	1.67	1.63	1.56	1.51	1.46	1.40	1.34	1.30
750	1.80	1.72	1.66	1.62	1.55	1.50	1.45	1.39	1.33	1.29
1000	1.79	1.72	1.66	1.61	1.54	1.50	1.44	1.38	1.32	1.28

TABLE A.3 (continued)

F Distribution: Critical Values of F (0.1% significance level)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	4.05e05	5.00e05	5.40e05	5.62e05	5.76e05	5.86e05	5.93e05	5.98e05	6.02e05	6.06e05	6.11e05	6.14e05	6.17e05	6.19e05	6.21e05
2	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39	999.40	999.42	999.43	999.44	999.44	999.45
3	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86	129.25	128.32	127.64	127.14	126.74	126.42
4	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47	48.05	47.41	46.95	46.60	46.32	46.10
5	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24	26.92	26.42	26.06	25.78	25.57	25.39
6	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69	18.41	17.99	17.68	17.45	17.27	17.12
7	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33	14.08	13.71	13.43	13.23	13.06	12.93
8	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77	11.54	11.19	10.94	10.75	10.60	10.48
9	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11	9.89	9.57	9.33	9.15	9.01	8.90
10	21.04	14.91	12.55	11.28	10.48	9.93	9.52	9.20	8.96	8.75	8.45	8.22	8.05	7.91	7.80
11	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12	7.92	7.63	7.41	7.24	7.11	7.01
12	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48	7.29	7.00	6.79	6.63	6.51	6.40
13	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98	6.80	6.52	6.31	6.16	6.03	5.93
14	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58	6.40	6.13	5.93	5.78	5.66	5.56
15	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26	6.08	5.81	5.62	5.46	5.35	5.25
16	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98	5.81	5.55	5.35	5.20	5.09	4.99
17	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75	5.58	5.32	5.13	4.99	4.87	4.78
18	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56	5.39	5.13	4.94	4.80	4.68	4.59
19	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39	5.22	4.97	4.78	4.64	4.52	4.43
20	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24	5.08	4.82	4.64	4.49	4.38	4.29
21	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11	4.95	4.70	4.51	4.37	4.26	4.17
22	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99	4.83	4.58	4.40	4.26	4.15	4.06
23	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89	4.73	4.48	4.30	4.16	4.05	3.96
24	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80	4.64	4.39	4.21	4.07	3.96	3.87
25	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71	4.56	4.31	4.13	3.99	3.88	3.79
26	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64	4.48	4.24	4.06	3.92	3.81	3.72
27	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57	4.41	4.17	3.99	3.86	3.75	3.66
28	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50	4.35	4.11	3.93	3.80	3.69	3.60
29	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45	4.29	4.05	3.88	3.74	3.63	3.54
30	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39	4.24	4.00	3.82	3.69	3.58	3.49
35	12.90	8.47	6.79	5.88	5.30	4.89	4.59	4.36	4.18	4.03	3.79	3.62	3.48	3.38	3.29
40	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02	3.87	3.64	3.47	3.34	3.23	3.14
50	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82	3.67	3.44	3.27	3.41	3.04	2.95
60	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69	3.54	3.32	3.15	3.02	2.91	2.83
70	11.80	7.64	6.06	5.20	4.66	4.28	3.99	3.77	3.60	3.45	3.23	3.06	2.93	2.83	2.74
80	11.67	7.54	5.97	5.12	4.58	4.20	3.92	3.70	3.53	3.39	3.16	3.00	2.87	2.76	2.68
90	11.57	7.47	5.91	5.06	4.53	4.15	3.87	3.65	3.48	3.34	3.11	2.95	2.82	2.71	2.63
100	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.44	3.30	3.07	2.91	2.78	2.68	2.59
120	11.38	7.32	5.78	4.95	4.42	4.04	3.77	3.55	3.38	3.24	3.02	2.85	2.72	2.62	2.53
150	11.27	7.24	5.71	4.88	4.35	3.98	3.71	3.49	3.32	3.18	2.96	2.80	2.67	2.56	2.48
200	11.15	7.15	5.63	4.81	4.29	3.92	3.65	3.43	3.26	3.12	2.90	2.74	2.61	2.51	2.42
250	11.09	7.10	5.59	4.77	4.25	3.88	3.61	3.40	3.23	3.09	2.87	2.71	2.58	2.48	2.39
300	11.04	7.07	5.56	4.75	4.22	3.86	3.59	3.38	3.21	3.07	2.85	2.69	2.56	2.46	2.37
400	10.99	7.03	5.53	4.71	4.19	3.83	3.56	3.35	3.18	3.04	2.82	2.66	2.53	2.43	2.34
500	10.96	7.00	5.51	4.69	4.18	3.81	3.54	3.33	3.16	3.02	2.81	2.64	2.52	2.41	2.33
600	10.94	6.99	5.49	4.68	4.16	3.80	3.53	3.32	3.15	3.01	2.80	2.63	2.51	2.40	2.32
750	10.91	6.97	5.48	4.67	4.15	3.79	3.52	3.31	3.14	3.00	2.78	2.62	2.49	2.39	2.31
1000	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	3.13	2.99	2.77	2.61	2.48	2.38	2.30

TABLE A.3 (continued)

F Distribution: Critical Values of F (0.1% significance level)

ν_2 \ ν_1	25	30	35	40	50	60	75	100	150	200
1	6.24e05	6.26e05	6.28e05	6.29e05	6.30e05	6.31e05	6.32e05	6.33e05	6.35e05	6.35e05
2	999.46	999.47	999.47	999.47	999.48	999.48	999.49	999.49	999.49	999.49
3	125.84	125.45	125.17	124.96	124.66	124.47	124.27	124.07	123.87	123.77
4	45.70	45.43	45.23	45.09	44.88	44.75	44.61	44.47	44.33	44.26
5	25.08	24.87	24.72	24.60	24.44	24.33	24.22	24.12	24.01	23.95
6	16.85	16.67	16.54	16.44	16.31	16.21	16.12	16.03	15.93	15.89
7	12.69	12.53	12.41	12.33	12.20	12.12	12.04	11.95	11.87	11.82
8	10.26	10.11	10.00	9.92	9.80	9.73	9.65	9.57	9.49	9.45
9	8.69	8.55	8.46	8.37	8.26	8.19	8.11	8.04	7.96	7.93
10	7.60	7.47	7.37	7.30	7.19	7.12	7.05	6.98	6.91	6.87
11	6.81	6.68	6.59	6.52	6.42	6.35	6.28	6.21	6.14	6.10
12	6.22	6.09	6.00	5.93	5.83	5.76	5.70	5.63	5.56	5.52
13	5.75	5.63	5.54	5.47	5.37	5.30	5.24	5.17	5.10	5.07
14	5.38	5.25	5.17	5.10	5.00	4.94	4.87	4.81	4.74	4.71
15	5.07	4.95	4.86	4.80	4.70	4.64	4.57	4.51	4.44	4.41
16	4.82	4.70	4.61	4.54	4.45	4.39	4.32	4.26	4.19	4.16
17	4.60	4.48	4.40	4.33	4.24	4.18	4.11	4.05	3.98	3.95
18	4.42	4.30	4.22	4.15	4.06	4.00	3.93	3.87	3.80	3.77
19	4.26	4.14	4.06	3.99	3.90	3.84	3.78	3.71	3.65	3.61
20	4.12	4.00	3.92	3.86	3.77	3.70	3.64	3.58	3.51	3.48
21	4.00	3.88	3.80	3.74	3.64	3.58	3.52	3.46	3.39	3.36
22	3.89	3.78	3.70	3.63	3.54	3.48	3.41	3.35	3.28	3.25
23	3.79	3.68	3.60	3.53	3.44	3.38	3.32	3.25	3.19	3.16
24	3.71	3.59	3.51	3.45	3.36	3.29	3.23	3.17	3.10	3.07
25	3.63	3.52	3.43	3.37	3.28	3.22	3.15	3.09	3.03	2.99
26	3.56	3.44	3.36	3.30	3.21	3.15	3.08	3.02	2.95	2.92
27	3.49	3.38	3.30	3.23	3.14	3.08	3.02	2.96	2.89	2.86
28	3.43	3.32	3.24	3.18	3.09	3.02	2.96	2.90	2.83	2.80
29	3.38	3.27	3.18	3.12	3.03	2.97	2.91	2.84	2.78	2.74
30	3.33	3.22	3.13	3.07	2.98	2.92	2.86	2.79	2.73	2.69
35	3.13	3.02	2.93	2.87	2.78	2.72	2.66	2.59	2.52	2.49
40	2.98	2.87	2.79	2.73	2.64	2.57	2.51	2.44	2.38	2.34
50	2.79	2.68	2.60	2.53	2.44	2.38	2.31	2.25	2.18	2.14
60	2.67	2.55	2.47	2.41	2.32	2.25	2.19	2.12	2.05	2.01
70	2.58	2.47	2.39	2.32	2.23	2.16	2.10	2.03	1.95	1.92
80	2.52	2.41	2.32	2.26	2.16	2.10	2.03	1.96	1.89	1.85
90	2.47	2.36	2.27	2.21	2.11	2.05	1.98	1.91	1.83	1.79
100	2.43	2.32	2.24	2.17	2.08	2.01	1.94	1.87	1.79	1.75
120	2.37	2.26	2.18	2.11	2.02	1.95	1.88	1.81	1.73	1.68
150	2.32	2.21	2.12	2.06	1.96	1.89	1.82	1.74	1.66	1.62
200	2.26	2.15	2.07	2.00	1.90	1.83	1.76	1.68	1.60	1.55
250	2.23	2.12	2.03	1.97	1.87	1.80	1.72	1.65	1.56	1.51
300	2.21	2.10	2.01	1.94	1.85	1.78	1.70	1.62	1.53	1.48
400	2.18	2.07	1.98	1.92	1.82	1.75	1.67	1.59	1.50	1.45
500	2.17	2.05	1.97	1.90	1.80	1.73	1.65	1.57	1.48	1.43
600	2.16	2.04	1.96	1.89	1.79	1.72	1.64	1.56	1.46	1.41
750	2.15	2.03	1.95	1.88	1.78	1.71	1.63	1.55	1.45	1.40
1000	2.14	2.02	1.94	1.87	1.77	1.69	1.62	1.53	1.44	1.38

TABLE A.4

χ^2 (Chi-Squared) Distribution: Critical Values of χ^2

	Significance level		
Degrees of freedom	5%	1%	0.1%
1	3.841	6.635	10.828
2	5.991	9.210	13.816
3	7.815	11.345	16.266
4	9.488	13.277	18.467
5	11.070	15.086	20.515
6	12.592	16.812	22.458
7	14.067	18.475	24.322
8	15.507	20.090	26.124
9	16.919	21.666	27.877
10	18.307	23.209	29.588

FORMULA SHEET

$$\widehat{\tau}_{hh} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{p_i}.$$

$$\mathrm{var}(\widehat{\tau}_{hh}) = \frac{1}{n}\sum_{i=1}^{N}p_i\left(\frac{y_i}{p_i}-\tau\right)^2$$

$$\widehat{\mathrm{var}}(\widehat{\tau}_{hh}) = \frac{1}{n(n-1)}\sum_{i=1}^{n}p_i\left(\frac{y_i}{p_i}-\widehat{\tau}_{hh}\right)^2$$

$$\widehat{\mu}_{hh} = \frac{1}{Nn}\sum_{i=1}^{n}\frac{y_i}{p_i}$$

$$\widehat{\mathrm{var}}(\widehat{\mu}_{hh}) = \frac{1}{N^2n(n-1)}\sum_{i=1}^{n}p_i\left(\frac{y_i}{p_i}-\widehat{\tau}_{hh}\right)^2$$

$$\widehat{\tau}_{hh} \pm z^*\sqrt{\mathrm{var}(\widehat{\tau}_{hh})}$$

$$\widehat{\mu}_{hh} \pm z^*\sqrt{\mathrm{var}(\widehat{\mu}_{hh})}$$

$$\widehat{\tau}_{ht} = \sum_{i=1}^{\nu}\frac{y_i}{\pi_i}$$

$$\widehat{\mu}_{ht} = \frac{1}{N}\sum_{i=1}^{\nu}\frac{y_i}{\pi_i}$$

$$\mathrm{var}(\widehat{\tau}_{ht}) = \sum_{i=1}^{N}\left(\frac{1}{\pi_i}-1\right)y_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\pi_{ij}}{\pi_i\pi_j}-1\right)y_iy_j$$

$$\widehat{\mathrm{var}}(\widehat{\tau}_{ht}) = \sum_{i=1}^{\nu}\left(\frac{1}{\pi_i^2}-\frac{1}{\pi_i}\right)y_i^2 + 2\sum_{i=1}^{\nu}\sum_{j>i}^{\nu}\left(\frac{1}{\pi_i\pi_j}-\frac{1}{\pi_{ij}}\right)y_iy_j$$

$$\widehat{\mathrm{var}}(\widehat{\mu}_{ht}) = \frac{1}{N^2}\widehat{\mathrm{var}}(\widehat{\tau}_{ht})$$

$$\widehat{\mathrm{var}}_{syg}(\widehat{\tau}_{ht}) = \sum^{\nu}\sum_{j>i}^{\nu}\left(\frac{y_i}{\pi_i}-\frac{y_j}{\pi_j}\right)^2\left(\frac{\pi_i\pi_j}{\pi_{ij}}-1\right)$$

$$\widehat{\text{var}}(\hat{\tau}_{bh}) = \frac{N-\nu}{N}\frac{s_t^2}{\nu}$$

$$s_t^2 = \frac{1}{(\nu-1)}\sum_{i=1}^{\nu}(t_i - \hat{\tau}_{ht})^2$$

$$t_i = \frac{\nu y_i}{\pi_i}$$

$$\hat{\mu}_{ht} \pm z^* \sqrt{\widehat{var}(\hat{\mu}_{ht})}$$

$$\hat{\tau}_{ht} \pm z^* \sqrt{\widehat{var}(\hat{\tau}_{ht})}$$

$$\hat{\mu}_{ht} \pm t^* \sqrt{\widehat{var}(\hat{\mu}_{ht})}$$

$$\hat{\tau}_{ht} \pm t^* \sqrt{\widehat{var}(\hat{\tau}_{ht})}$$

$$\tau = \sum_{i=1}^{N} y_i$$

$$\mu = \frac{1}{N}\sum_{i=1}^{N} y_i = \tau/N$$

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \mu)^2$$

$$\sigma^2 = \left(\frac{1}{N-1}\right)\left(\sum_{i=1}^{N} y_i^2 - \frac{\tau^2}{N}\right) = \left(\frac{1}{N-1}\right)\left(\sum_{i=1}^{N} y_i^2 - N\mu^2\right)$$

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} y_i^2 - \frac{(\sum_{i=1}^{n} y_i)^2}{n}\right)$$

$$\hat{\mu} = \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\hat{\tau} = \frac{N}{n}\sum_{i=1}^{n} y_i = N\overline{y}.$$

$$\mathrm{var}(\hat{\mu}) = \mathrm{var}(\overline{y}) = \left(\frac{N-n}{N}\right)\frac{\sigma^2}{n}$$

$$\mathrm{var}(\hat{\tau}) = N(N-n)\frac{\sigma^2}{n}$$

$$\widehat{\mathrm{var}}(\hat{\mu}) = \widehat{\mathrm{var}}(\overline{y}) = \left(\frac{N-n}{N}\right)\frac{s^2}{n}$$

$$\widehat{\mathrm{var}}(\hat{\tau}) = N(N-n)\frac{s^2}{n}$$

$$\overline{y} \pm z^*\frac{s}{\sqrt{n}}$$

$$\overline{y} \pm t^*\frac{s}{\sqrt{n}}$$

$$\overline{y} \pm z^*\sqrt{\left(\frac{N-n}{N}\right)\frac{s^2}{n}}$$

$$N\overline{y} \pm z^*\sqrt{N(N-n)\frac{s^2}{n}}$$

$$\overline{y} \pm t^*\sqrt{\left(\frac{N-n}{N}\right)\frac{s^2}{n}}$$

$$N\overline{y} \pm t^*\sqrt{N(N-n)\frac{s^2}{n}}$$

$$n = \frac{1}{\frac{d^2}{z^2\sigma^2} + \frac{1}{N}} = \frac{1}{\frac{1}{n_0} + \frac{1}{N}}$$

$$n_0 = \frac{z^2\sigma^2}{d^2}$$

$$n = \frac{1}{\frac{d^2}{N^2z^2\sigma^2} + \frac{1}{N}} = \frac{1}{\frac{1}{n_0} + \frac{1}{N}}$$

$$n_0 = \frac{N^2z^2\sigma^2}{d^2}$$

$$\widehat{p} = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - p)^2 = \left(\frac{N}{N-1}\right) p(1-p)$$

$$\text{var}(\widehat{p}) = \left(\frac{N-n}{N}\right)\frac{\sigma^2}{n}$$

$$\widehat{\text{var}}(\widehat{p}) = \left(\frac{N-n}{N}\right)\frac{s^2}{n} = \left(\frac{N-n}{N}\right)\frac{\widehat{p}(1-\widehat{p})}{n-1}$$

$$\widehat{p} \pm z^* \sqrt{\widehat{\text{var}}(\widehat{p})}$$

$$n = \frac{Np(1-p}{(N-1)\frac{d^2}{z^2} + p(1-p)} = \frac{1}{\frac{N-1}{Nn_0} + \frac{1}{N}} \approx \frac{1}{\frac{1}{n_0} + \frac{1}{N}}$$

$$n_0 = \frac{z^2 p(1-p)}{d^2}$$

$$R = \frac{\mu_y}{\mu_x} = \frac{\tau_y}{\tau_x}$$

$$\mu_x = \left(\sum_{i=1}^{N} x_i\right)/N \qquad \tau_x = \sum_{i=1}^{N} x_i \qquad \mu_y = \left(\sum_{i=1}^{N} y_i\right)/N \qquad \tau_y = \sum_{i=1}^{N} y_i$$

$$\overline{x} = \sum_{i=1}^{n} x_i/n$$

$$\overline{y} = \sum_{i=1}^{n} y_i/n$$

$$r = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i} = \frac{\overline{y}}{\overline{x}}$$

$$\widehat{\text{Var}}(r) = \left(\frac{N-n}{N\mu_x^2}\right)\frac{s_r^2}{n} = \left(\frac{N(N-n)}{\tau_x^2}\right)\frac{s_r^2}{n}$$

$$s_r^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - rx_i)^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} y_i^2 + r^2\sum_{i=1}^{n} x_i^2 - 2r\sum_{i=1}^{n} x_i y_i\right)$$

$$\widehat{\mu}_r = r\mu_x$$

$$\widehat{\text{Var}}(\widehat{\mu}_r) = \left(\frac{N-n}{N}\right)\frac{s_r^2}{n}$$

$$\hat{\tau}_r = Nr\mu_x = r\tau_x$$

$$\widehat{\mathrm{Var}}(\hat{\tau}_r) = N(N-n)\frac{s_r^2}{n}$$

$$\widetilde{\mathrm{Var}}(\hat{\mu}_r) = \left(\frac{\mu_x^2}{\overline{x}^2}\right)\widehat{\mathrm{Var}}(\hat{\mu}_r)$$

$$\widetilde{\mathrm{Var}}(\hat{\tau}_r) = \left(\frac{\mu_x^2}{\overline{x}^2}\right)\widehat{\mathrm{Var}}(\hat{\tau}_r)$$

$$r \pm z^*\sqrt{\widehat{\mathrm{Var}}(r)} \qquad \hat{\mu}_r \pm z^*\sqrt{\widehat{\mathrm{Var}}(\hat{\mu}_r)} \qquad \hat{\tau}_r \pm z^*\sqrt{\widehat{\mathrm{Var}}(\hat{\tau}_r)}$$

$$r \pm t^*\sqrt{\widehat{\mathrm{Var}}(r)} \qquad \hat{\mu}_r \pm t^*\sqrt{\widehat{\mathrm{Var}}(\hat{\mu}_r)} \qquad \hat{\tau}_r \pm t^*\sqrt{\widehat{\mathrm{Var}}(\hat{\tau}_r)}$$

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_y\sigma_x}$$

$$\sigma_{xy} = \frac{\sum_{i=1}^{N}(x_i-\mu_x)(y_i-\mu_y)}{N-1}$$

$$\mathrm{Var}(\hat{\tau}_r) = \frac{N(N-n)}{n}\left(\sigma_y^2 - 2R\rho_{xy}\sigma_y\sigma_x + R^2\sigma_x^2\right)$$

$$\mathrm{Var}(\hat{\mu}_r) = \frac{N-n}{Nn}\left(\sigma_y^2 - 2R\rho_{xy}\sigma_y\sigma_x + R^2\sigma_x^2\right)$$

$$\hat{C}_x = \frac{s_x}{\overline{x}} \qquad \hat{C}_y = \frac{s_y}{\overline{y}}$$

$$\hat{\rho} = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{s_x}\right)\left(\frac{y_i-\overline{y}}{s_y}\right) = \frac{\sum_{i=1}^n x_iy_i - \frac{1}{n}(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{(n-1)s_xs_y}$$

$$a = \overline{y} - b\overline{x} \qquad b = \frac{\sum_{i=1}^n(x_i-\overline{x})(y_i-\overline{y})}{\sum_{i=1}^n(x_i-\overline{x})^2} = \frac{n\sum_{i=1}^n x_iy_i - (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{n\sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2}$$

$$\hat{\mu}_L = a + b\mu_x = (\overline{y} - b\overline{x}) + b\mu_x = \overline{y} + b(\mu_x - \overline{x})$$

$$\widehat{\mathrm{var}}(\hat{\mu}_L) = \frac{N-n}{Nn(n-2)}\sum_{i=1}^{n}(y_i - a - bx_i)^2$$

$$\widehat{\mathrm{var}}(\hat{\mu}_L) \approx \frac{1}{n(n-2)}\sum_{i=1}^{n}(y_i - a - bx_i)^2$$

$$\widehat{\text{var}}(\hat{\mu}_L) = \frac{N-n}{Nn(n-2)}\left[\sum_{i=1}^{n} y_i^2 - n\overline{y}^2 - b^2\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)\right]$$

$$\hat{\tau}_L = N\hat{\mu}_L = N(a + b\mu_x) = N\overline{y} + b(\tau_x - N\overline{x})$$

$$\widehat{\text{var}}(\hat{\tau}_L) = \frac{N(N-n)}{n(n-2)}\sum_{i=1}^{n}(y_i - a - bx_i)^2$$

$$\widehat{\text{var}}(\hat{\mu}_L) = \frac{N-n}{Nn(n-2)}SSE = \frac{N-n}{Nn}MSE$$

$$\widehat{\text{var}}(\hat{\tau}_L) = \frac{N(N-n)}{n(n-2)}SSE = \frac{N(N-n)}{n}MSE$$

$$\hat{y} = b_0 + \sum_{i=1}^{k} b_i x_i$$

$$\hat{\mu}_L = b_0 + \sum_{i=1}^{k} b_i \mu_i$$

$$\hat{y} = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k$$

$$\hat{\mu}_L = b_0 + b_1\mu_x + b_2\mu_x^2 + \cdots + b_k\mu_x^k$$

$$\widehat{\text{var}}(\hat{\mu}_L) = \frac{N-n}{Nn(n-k-1)}SSE = \frac{N-n}{Nn}MSE$$

$$z_{\alpha/2}\sqrt{\widehat{\text{Var}}(r)} = z_{\alpha/2}\sqrt{\left(\frac{N-n}{N\mu_x^2}\right)\frac{\sigma_r^2}{n}}$$

$$z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\hat{\mu}_r)} = z_{\alpha/2}\sqrt{\left(\frac{N-n}{N}\right)\frac{\sigma_r^2}{n}}$$

$$z_{\alpha/2}\sqrt{\widehat{\text{Var}}(\hat{\tau}_r)} = z_{\alpha/2}\sqrt{N(N-n)\frac{\sigma_r^2}{n}}$$

For R : $n = \dfrac{1}{\frac{1}{n_0} + \frac{1}{N}}$ where $n_0 = \dfrac{z_{\alpha/2}^2\,\hat{\sigma}_r^2}{d^2\mu_x^2}$

$$\text{For } \mu_y : \quad n = \frac{1}{\frac{1}{n_0} + \frac{1}{N}} \quad \text{where } n_0 = \frac{z^2_{\alpha/2}\,\widehat{\sigma}^2_r}{d^2}$$

$$\text{For } \tau_y : \quad n = \frac{1}{\frac{1}{n_0} + \frac{1}{N}} \quad \text{where } n_0 = \frac{N^2 z^2_{\alpha/2}\,\widehat{\sigma}^2_r}{d^2}$$

$$\text{For } \mu_y : \quad n = \frac{1}{\frac{1}{n_0} + \frac{1}{N}} \quad \text{where } n_0 = \frac{z^2_{\alpha/2}\,\widehat{\sigma}^2_{regr}}{d^2}$$

$$\text{For } \tau_y : \quad n = \frac{1}{\frac{1}{n_0} + \frac{1}{N}} \quad \text{where } n_0 = \frac{N^2 z^2_{\alpha/2}\,\widehat{\sigma}^2_{regr}}{d^2}$$

$$\tau_{y(d)} = \sum_{i \in U_d} y_i \qquad \mu_{y(d)} = \tau_{y(d)}/N_d = \left(\sum_{i \in U_d} y_i\right)/N_d$$

$$\widehat{\mu}_{y(d)} = \left(\sum_{i \in S_d} y_i\right)/n_d = \overline{y}_d \qquad \widehat{\tau}_{y(d)} = \frac{N_d}{n_d}\left(\sum_{i \in S_d} y_i\right) = N_d \overline{y}_d$$

$$\widehat{\text{Var}}(\overline{y}_d) = \widehat{\text{Var}}(r_d) = \left(\frac{N-n}{N\mu^2_{x(d)}}\right)\frac{s^2_r}{n} = \left(\frac{N-n}{N(N_d/N)^2}\right)\frac{s^2_r}{n} = \left(\frac{N-n}{N}\right)\left(\frac{N}{N_d}\right)^2 \frac{s^2_r}{n}$$

$$s^2_r = \frac{1}{n-1}\left(\sum_{i \in S_d} y_i^2 + n_d r_d^2 - 2r_d \sum_{i \in S_d} y_i\right)$$

$$\tau_h = \sum_{i=1}^{N_h} y_{hi} \qquad \tau = \sum_{h=1}^{L}\sum_{i=1}^{N_h} y_{hi} = \sum_{h=1}^{L} \tau_h$$

$$\mu_h = \frac{\tau_h}{N_h} \qquad \mu = \frac{1}{N}\sum_{h=1}^{L}\sum_{i=1}^{N_h} y_{hi}$$

$$\widehat{\mu}_h = \overline{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi} \qquad \widehat{\tau}_h = N_h \overline{y}_h = \frac{N_h}{n_h}\sum_{i=1}^{n_h} y_{hi}$$

$$\widehat{\mu}_{st} = \frac{\widehat{\tau}_{st}}{N} = \frac{1}{N}\sum_{h=1}^{L} N_h \overline{y}_h \qquad \widehat{\mu}_{st} = \sum_{h=1}^{L} W_h \overline{y}_h$$

$$\hat{\tau}_{st} = \sum_{h=1}^{L} \hat{\tau}_h = \sum_{h=1}^{L} N_h \overline{y}_h$$

$$\text{var}(\hat{\mu}_h) = \left(\frac{N_h - n_h}{N_h}\right) \frac{\sigma_h^2}{n_h} \qquad \text{var}(\hat{\tau}_h) = N_h(N_h - n_h)\frac{\sigma_h^2}{n_h}$$

$$\sigma_h^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hi} - \mu_h)^2$$

$$\text{var}(\hat{\tau}_{st}) = \sum_{i=1}^{L} \text{var}(\hat{\tau}_h) = \sum_{i=1}^{L} N_h(N_h - n_h)\frac{\sigma_h^2}{n_h}$$

$$\text{var}(\hat{\mu}_{st}) = \left(\frac{1}{N^2}\right) \text{var}(\hat{\tau}_{st}) = \left(\frac{1}{N^2}\right) \sum_{i=1}^{L} N_h(N_h - n_h)\frac{\sigma_h^2}{n_h}$$

$$\widehat{\text{var}}(\hat{\tau}_{st}) = \sum_{h=1}^{L} N_h(N_h - n_h)\frac{s_h^2}{n_h} \qquad \widehat{\text{var}}(\hat{\mu}_{st}) = \left(\frac{1}{N^2}\right) \sum_{h=1}^{L} N_h(N_h - n_h)\frac{s_h^2}{n_h}$$

$$\hat{\mu}_{st} \pm z^* \sqrt{\widehat{\text{var}}(\hat{\mu}_{st})} \qquad \hat{\tau}_{st} \pm z^* \sqrt{\widehat{\text{var}}(\hat{\tau}_{st})}$$

$$\hat{\mu}_{st} \pm t^* \sqrt{\widehat{\text{var}}(\hat{\mu}_{st})} \qquad \hat{\tau}_{st} \pm t^* \sqrt{\widehat{\text{var}}(\hat{\tau}_{st})}$$

$$d = \frac{\left(\sum_{h=1}^{L} a_h s_h^2\right)^2}{\sum_{h=1}^{L} (a_h s_h^2)^2/(n_h - 1)} = \frac{(\widehat{\text{var}}(\hat{\tau}_{st}))^2}{\sum_{h=1}^{L} (a_h s_h^2)^2/(n_h - 1)}$$

$$n_h = \frac{(C - c_0) N_h \sigma_h / \sqrt{c_h}}{\sum_{h=1}^{L} N_h \sigma_h \sqrt{c_h}}$$

$$p_h = \frac{1}{N_h} \sum_{i=1}^{N_h} y_{hi} \qquad \tau = \sum_{h=1}^{L} \sum_{i=1}^{N_h} y_{hi}$$

$$\hat{p}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi} \qquad \hat{p}_{st} = \sum_{h=1}^{L} \frac{N_h}{N} \hat{p}_h = \sum_{h=1}^{L} W_h \hat{p}_h$$

$$\text{var}(\hat{p}_{st}) = \sum_{h=1}^{L} W_h^2 \,\text{var}(\hat{p}_h) = \sum_{h=1}^{L} W_h^2 \left(\frac{N_h - n_h}{N_h}\right) \frac{\sigma_h^2}{n_h}$$

$$\widehat{\text{var}}(\hat{p}_{st}) = \sum_{h=1}^{L} W_h^2 \left(\frac{N_n - n_h}{N_h}\right) \frac{s_h^2}{n_h} = \sum_{h=1}^{L} W_h^2 \frac{N_h - n_h}{N_h} \frac{\hat{p}_h(1 - \hat{p}_h)}{n_h - 1}$$

$$\widehat{p}_{st} \pm t^* \sqrt{\widehat{\mathrm{var}}(\widehat{p}_{st})}$$

$$\widehat{\mu}_h = \frac{\overline{y}_h}{\overline{x}_h}\mu_{h(x)} = r_h \mu_{h(x)}$$

$$\widehat{\mu}_{sst} = \sum_{h=1}^{L} W_h r_h \mu_{h(x)} = \sum_{h=1}^{L} \frac{N_h}{N}\frac{\overline{y}_h}{\overline{x}_h}\mu_{h(x)}$$

$$\begin{aligned}\widehat{\mathrm{var}}(\widehat{\mu}_{sst}) &= \sum_{h=1}^{L}\left[\left(\frac{N_h}{N}\right)^2 \frac{N_h - n_h}{N_h n_h (n_h - 1)}\left(\sum_{i=1}^{n_h} y_{hi}^2 + r_h^2 \sum_{i=1}^{n_h} x_{hi}^2 - 2r_h \sum_{i=1}^{n_h} x_{hi}y_{hi}\right)\right] \\ &= \sum_{h=1}^{L}\left[W_h^2 \frac{1 - f_h}{n_h(n_h - 1)}\left(\sum_{i=1}^{n_h} y_{hi}^2 + r_h^2 \sum_{i=1}^{n_h} x_{hi}^2 - 2r_h \sum_{i=1}^{n_h} x_{hi}y_{hi}\right)\right]\end{aligned}$$

$$\widehat{\mu}_{sst} \pm z^* \sqrt{\widehat{\mathrm{var}}(\widehat{\mu}_{sst})} \qquad \widehat{\mu}_{sst} \pm t^* \sqrt{\widehat{\mathrm{var}}(\widehat{\mu}_{sst})}$$

$$\overline{x}_{st} = \sum_{h=1}^{L} \frac{N_h}{N}\overline{x}_h \qquad \overline{y}_{st} = \sum_{h=1}^{L} \frac{N_h}{N}\overline{y}_h \qquad r_c = \frac{\overline{y}_{st}}{\overline{x}_{st}}$$

$$\begin{aligned}\widehat{\mathrm{var}}(\widehat{\mu}_{cst}) &= \sum_{h=1}^{L}\left[W_h^2 \frac{(1 - f_h)}{n_h(n_h - 1)}\left(\sum_{i=1}^{n_h} y_{hi}^2 + r_c^2 \sum_{i=1}^{n_h} x_{hi}^2 - 2r_c \sum_{i=1}^{n_h} x_{hi}y_{hi}\right)\right] \\ &= \sum_{h=1}^{L}\left[\left(\frac{N_h}{N}\right)^2 \frac{(1 - f_h)}{n_h(n_h - 1)}\left(\sum_{i=1}^{n_h} y_{hi}^2 + r_c^2 \sum_{i=1}^{n_h} x_{hi}^2 - 2r_c \sum_{i=1}^{n_h} x_{hi}y_{hi}\right)\right]\end{aligned}$$

$$\widehat{y} = a_h + b_h x$$

$$b_h = \frac{n_h \sum_{i=1}^{n_h} x_{hi}y_{hi} - (\sum_{i=1}^{n_h} x_{hi})(\sum_{i=1}^{n_h} y_{hi})}{n_h \sum_{i=1}^{n_h} x_{hi}^2 - (\sum_{i=1}^{n_h} x_{hi})^2}$$

$$\begin{aligned}\widehat{\mu}_{Lh} &= a_h + b_h \mu_{h(x)} \\ &= \overline{y}_h + b_h(\mu_{h(x)} - \overline{x}_h)\end{aligned}$$

$$
\begin{aligned}
\widehat{\mu}_{Lsst} &= \sum_{h=1}^{L} \frac{N_h}{N} \left(a_h + b_h \mu_{h(x)}\right) \\
&= \sum_{h=1}^{L} \frac{N_h}{N} \left(\overline{y}_h + b_h(\mu_{h(x)} - \overline{x}_h)\right)
\end{aligned}
$$

$$
\widehat{\text{var}}(\widehat{\mu}_{Lsst}) = \sum_{h=1}^{L} \left(\frac{N_h}{N}\right)^2 \frac{(1-f_h)}{n_h(n_h-2)} SSE_h = \sum_{h=1}^{L} \left(\frac{N_h}{N}\right)^2 \frac{(1-f_h)}{n_h} MSE_h
$$

$$
\widehat{\tau}_{Lsst} = N\widehat{\mu}_{Lsst} \qquad \widehat{\text{var}}(\widehat{\tau}_{Lsst}) = N^2 \widehat{\text{var}}(\widehat{\mu}_{Lsst})
$$

$$
b_c = \frac{\sum_{h=1}^{L} c_h b_h}{\sum_{h=1}^{L} c_h}
$$

$$
\mu_{Lcst} = \widehat{\mu}_{st} + b_c(\mu_x - \overline{x}_{st})
$$

$$
c_h = \left(\frac{N_h}{N}\right)^2 \frac{(1-f_h)}{n_h} s_{xh}^2
$$

$$
\mu_{Lcst} = \widehat{\mu}_{st} + b_c(\mu_x - \overline{x}_{st})
$$

$$
\widehat{\text{var}}(\mu_{Lcst}) = \sum_{h=1}^{L} \left[\left(\frac{N_h}{N}\right)^2 \frac{(1-f_h)}{n_h(n_h-2)} \sum_{i=1}^{n_h} [(y_{hi} - \overline{y}_h) - b_c(x_{hi} - \overline{x}_h)]^2\right]
$$

$$
\widehat{\mu}_{pst} = \frac{1}{N} \sum_{h=1}^{L} N_h \overline{y}_h \qquad \widehat{\tau}_{pst} = \sum_{h=1}^{L} N_h \overline{y}_h
$$

$$
\widehat{\text{var}}(\widehat{\mu}_{pst}) = \frac{N-n}{nN} \sum_{i=1}^{L} \left(\frac{N_h}{N}\right) s_h^2 + \frac{1}{n^2} \left(\frac{N-n}{N-1}\right) \sum_{i=1}^{L} \frac{N-N_h}{N} s_h^2
$$

$$
\widehat{\text{var}}(\widehat{\tau}_{pst}) = \frac{N-n}{n} \sum_{i=1}^{L} N_h s_h^2 + \left(\frac{N^2}{n^2}\right) \left(\frac{N-n}{N-1}\right) \sum_{i=1}^{L} \frac{N-N_h}{N} s_h^2
$$

$$
\overline{y}_i = \frac{y_i}{M_i} \qquad y_i = \sum_{j=1}^{M_i} y_{ij} \qquad s_u^2 = \frac{\sum_{i=1}^{n} (y_i - \overline{y})^2}{n-1}
$$

$$\tau = \sum_{i=1}^{N}\sum_{j=1}^{M_i} y_{ij} = \sum_{i=1}^{N} y_i \qquad \mu = \frac{1}{M}\sum_{i=1}^{N}\sum_{j=1}^{M_i} y_{ij} = \frac{\tau}{M}$$

$$\mu_1 = \frac{1}{N}\sum_{i=1}^{N} y_i \qquad \sigma_u^2 = \frac{\sum_{i=1}^{N}(y_i - \mu_1)^2}{N-1}$$

$$\hat{\tau}_{cl} = \frac{M}{nL}\sum_{i=1}^{n}\sum_{j=1}^{L} y_{ij} = \frac{N}{n}\sum_{i=1}^{n}\sum_{j=1}^{L} y_{ij} = \frac{N}{n}\sum_{i=1}^{n} y_i = N\overline{y}$$

$$\hat{\mu}_{cl} = \frac{1}{nL}\sum_{i=1}^{n}\sum_{j=1}^{L} y_{ij} = \frac{1}{nL}\sum_{i=1}^{n} y_i = \frac{\overline{y}}{L} = \frac{\hat{\tau}_{cl}}{M}$$

$$\text{var}(\hat{\tau}_{cl}) = N(N-n)\,\frac{\sigma_u^2}{n} \qquad \text{var}(\hat{\mu}_{cl}) = \frac{N(N-n)}{M^2}\,\frac{\sigma_u^2}{n}$$

$$\widehat{\text{var}}(\hat{\tau}_{cl}) = N(N-n)\,\frac{s_u^2}{n} \qquad \widehat{\text{var}}(\hat{\mu}_{cl}) = \frac{N(N-n)}{M^2}\,\frac{s_u^2}{n}$$

$$\hat{\mu}_{cl} \pm t^*\sqrt{\widehat{\text{var}}(\hat{\mu}_{cl})} \qquad \hat{\tau}_{cl} \pm t^*\sqrt{\widehat{\text{var}}(\hat{\tau}_{cl})}$$

$$\hat{\tau}_{sys} = \frac{N}{n}\sum_{i=1}^{n} y_i = N\overline{y} \qquad \hat{\mu}_{sys} = \frac{1}{nL}\sum_{i=1}^{n} y_i = \frac{\overline{y}}{L} = \frac{\hat{\tau}_{sys}}{M}$$

$$\widehat{\text{var}}(\hat{\tau}_{sys}) = N(N-n)\frac{s_u^2}{n} \qquad \widehat{\text{var}}(\hat{\mu}_{sys}) = \frac{N(N-n)}{M^2}\frac{s_u^2}{n}$$

$$\hat{\mu}_{sys} \pm t^*\sqrt{\widehat{\text{var}}(\hat{\mu}_{sys})} \qquad \hat{\tau}_{sys} \pm t^*\sqrt{\widehat{\text{var}}(\hat{\tau}_{sys})}$$

$$\mu = \frac{\sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} M_i} = \frac{\sum_{i=1}^{N} y_i}{M} \qquad \mu = \left(\frac{N}{M}\right)\sum_{i=1}^{N}\frac{y_i}{N}$$

$$\hat{\mu}_{c(a)} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} M_i} = \frac{\sum_{i=1}^{n} y_i}{m}$$

$$\widehat{\text{var}}(\hat{\mu}_{c(a)}) = \frac{(N-n)N}{n(n-1)M^2}\sum_{i=1}^{n} M_i^2\,(\overline{y}_i - \hat{\mu}_{c(a)})^2$$

$$\widehat{\text{var}}(\hat{\mu}_{c(a)}) = \frac{(N-n)n}{N(n-1)} \sum_{i=1}^{n} \left(\frac{M_i}{m}\right)^2 (\overline{y}_i - \hat{\mu}_{c(a)})^2$$

$$\hat{\mu}_{c(b)} = \frac{N}{M} \frac{\sum_{i=1}^{n} y_i}{n} = \frac{N}{nM} \sum_{i=1}^{n} y_i$$

$$\widehat{\text{var}}(\hat{\mu}_{c(b)}) = \frac{(N-n)N}{n(n-1)M^2} \sum_{i=1}^{n} (y_i - \overline{y})^2 = \frac{(N-n)N}{nM^2} s_u^2$$

$$\widehat{\text{var}}(\hat{\mu}_{c(b)}) = \frac{(N-n)n}{(n-1)Nm^2} \sum_{i=1}^{n} (y_i - \overline{y})^2 = \frac{(N-n)n}{Nm^2} s_u^2$$

$$\hat{p}_c = \frac{\sum_{i=1}^{n} p_i}{n}$$

$$\text{var}(\hat{p}_c) = \left(\frac{N-n}{nN}\right) \sum_{i=1}^{N} \frac{(p_i - p)^2}{N-1} = \left(\frac{1-f}{n}\right) \sum_{i=1}^{N} \frac{(p_i - p)^2}{N-1}$$

$$\widehat{\text{var}}(\hat{p}_c) = \left(\frac{N-n}{nN}\right) \sum_{i=1}^{n} \frac{(p_i - \hat{p}_c)^2}{n-1} = \left(\frac{1-f}{n}\right) \sum_{i=1}^{n} \frac{(p_i - \hat{p}_c)^2}{n-1}$$

$$\hat{p}_c = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} M_i}$$

$$\text{var}(\hat{p}_c) \approx = \left(\frac{1-f}{n\overline{M}^2}\right) \frac{\sum_{i=1}^{N} (y_i - pM_i)^2}{N-1}$$

$$\widehat{\text{var}}(\hat{p}_c) \approx \left(\frac{1-f}{n\overline{m}^2}\right) \frac{\sum_{i=1}^{n} (y_i - p_c M_i)^2}{n-1}$$

$$= \left(\frac{1-f}{n\overline{m}^2}\right) \frac{\sum_{i=1}^{n} y_i^2 - 2p_c \sum_{i=1}^{n} y_i M_i + p_c^2 \sum_{i=1}^{n} M_i^2}{n-1}$$

$$\hat{y}_i = \frac{M_i}{m_i} \sum_{j=1}^{m_i} y_{ij} = M_i \overline{y}_i \quad \hat{\tau} = \frac{N}{n} \sum_{i=1}^{n} \hat{y}_i \quad \hat{\mu}_1 = \frac{\sum_{i=1}^{n} \hat{y}_i}{n}$$

$$\mathrm{var}(\hat{\tau}) = N(N-n)\frac{\sigma_u^2}{n} + \frac{N}{n}\sum_{i=1}^{N} M_i(M_i - m_i)\frac{\sigma_i^2}{m_i}$$

$$\sigma_u^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \mu_1)^2 \quad \sigma_i^2 = \frac{1}{M_i - 1}\sum_{j=1}^{M_i}(y_{ij} - \mu_{i\cdot})^2$$

$$\widehat{\mathrm{var}}(\hat{\tau}) = N(N-n)\frac{s_u^2}{n} + \frac{N}{n}\sum_{i=1}^{n} M_i(M_i - m_i)\frac{s_i^2}{m_i}$$

$$s_u^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\hat{y}_i - \hat{\mu}_1)^2 \quad s_i^2 = \frac{1}{m_i - 1}\sum_{j=1}^{m_i}(y_{ij} - \overline{y}_i)^2$$

$$\hat{\tau} = \frac{M}{nl}\sum_{i=1}^{n}\sum_{j=1}^{l} y_{ij} = \frac{M}{n}\sum_{i=1}^{n}\overline{y}_i$$

$$\hat{\mu} = \frac{1}{nl}\sum_{i=1}^{n}\sum_{j=1}^{l} y_{ij} = \frac{1}{n}\sum_{i=1}^{n}\overline{y}_i$$

$$\widehat{\mathrm{var}}(\hat{\tau}) = N(N-n)\frac{s_u^2}{n} + \frac{M(L-l)}{nl}\sum_{i=1}^{n} s_i^2$$

$$\widehat{\mathrm{var}}(\hat{\mu}) = \frac{N(N-n)}{M^2}\frac{s_u^2}{n} + \frac{L-l}{Mnl}\sum_{i=1}^{n} s_i^2$$

$$\hat{\tau} \pm t^*\sqrt{\widehat{\mathrm{var}}(\hat{\tau})} \qquad \hat{\mu} \pm t^*\sqrt{\widehat{\mathrm{var}}(\hat{\mu})}$$