UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2010

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ATTEMPT ALL QUESTIONS

Question 1 [20 marks, 6+7+7]

(a) Suppose $\log(X) \sim Gamma(\alpha, \lambda)$, then the random variable X is said to have a Loggamma distribution, $x > 1$.

(i) Show that the *pdf* of X is given by:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}(\log(X))^{\alpha-1}x^{-\lambda-1}, \qquad x > 1$$

(ii) A random sample $x_1, x_2, \cdots, x_n$, from X is observed. Let $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$ denote the maximum likelihood estimates of α and λ , respectively, based on these data. Show that $\hat{\alpha}_{ML}$ satisfies the equation

$$n \log\left\{\frac{n\hat{\alpha}_{ML}}{\sum_{i=1}^{n}\log(x_i)}\right\} - n\varphi(\hat{\alpha}_{ML}) + \sum_{i=1}^{n}\log\left(\log[x_i]\right) = 0$$

where φ is the di-gamma function.

(b) Suppose $X|C \sim Weibull(c, \gamma)$ and $C \sim Gamma(\alpha, \lambda)$, $c > 0$. Derive the marginal distribution of X and identify the distribution.

Question 2 [20 marks, 3+5+5+5+2]

(a) Consider random variables X and Y with joint density function

$$f_{X,Y}(x,y) = \begin{cases} k(3x-2) & \text{for } 0 < y < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find k.

(ii) Find $f_X(x)$. Hence evaluate $E(X)$.

(iii) Evaluate $P(2Y > X)$.

(b) In a game, there are 5 covered boxes, of which 2 contain identical prizes and the remaining 3 are empty. First you choose a box without opening it. Then the host of the game, knowing which boxes contain the prizes, choose an empty box to reveal it to you. You are then given a chance to switch to one of the remaining 3 boxes, or stick with the originally chosen box. You win a prize if the box you choose in the end contains one.

(i) Suppose you will change to one of the 3 remaining boxes randomly. What is the probability that you win a prize at the end?

(ii) Now, suppose that you will not change to another box. What is the probability that you win a prize at the end?

Question 3 [20 marks, 4+2+7+5+2]

(a) You flip a coin which shows a head with probability p and a tail with probability $q = 1 - p$. If the first flip shows a head, you record the number of consecutive heads, X, you get from the second flip onwards. Then you record the number of consecutive tails, Y , you get after the consecutive run of heads is over. Similarly, if the first flip shows a tail, X denotes the consecutive number of tails from the second flip onwards, and Y denotes the number of consecutive heads after the run of consecutive tails is over. Show that the joint probability mass function of X and Y is

$$p_{X,Y}(x,y) = p^{x+2}q^y + q^{x+2}p^y, \quad x,y \text{ are integers}, \quad x \geq 0, y \geq 1.$$

(b) A biologist studies the number of offsprings, N, produced by a single bacteria until it dies. Suppose $N \sim Poisson(\mu)$. A disease can infect any individual offspring independently with probability p. Let N_d denote the number of infected offspring, and N_h denote the number of healthy offspring, so that $N_d + N_h = N$.

- (i) Write N_d as a sum of random variables, carefully defining the variables involved.
- (ii) Hence, derive the moment generating function of N_d, and find the mean and variance of N_d by differentiation of the MGF.

(c) Suppose that we know X and Y are independent and have continuous uniform distributions, $X \sim U(0,2)$ and $Y \sim U(1,3)$. Let $Z = X + Y$.

- (i) Derive the density of Z.
- (ii) Find $P(Z > 2)$.

Appendix

- A continuous non-negative random variable X is distributed Gamma(α,λ), with p.d.f.

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}\exp(-\lambda x)}{\Gamma(\alpha)}, \qquad x \geq 0; \quad \alpha, \lambda > 0$$

- The 2-parameter Weibull distribution has the c.d.f.

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{b}\right)^c\right\}, \qquad x \geq 0.$$

- The function

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt, \quad p > 0,$$

has the properties

$$\Gamma(p+1) = p\Gamma(p) : \quad \Gamma(1/2) = \sqrt{\pi} : \quad \Gamma(n+1) = n!, \qquad \text{n integer } \geq 0.$$