UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2011

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR AND STATISTICAL TABLES
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 **[20 marks, 3+9+8]**

Let X and Y be random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} ke^{-\lambda x}, & 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k.

(b) Derive the marginal density for Y and hence evaluate $E(Y)$, $E(Y^2)$ and $Var(Y)$.

(c) Derive the conditional density, $f_{X|Y}(x|y)$, and the conditional expectation, $E[X|Y]$. Hence or otherwise, evaluate $E(X)$ and $Cov(X,Y)$.

Question 2 **[20 marks, 3+3+3+2+5+4]**

(a) In 1995 an account on the UNISWA network came with a three letter (all uppercase roman letter) password. Suppose a malicious hacker could check one password every millisecond.

(i) Assuming the hacker knows a username and the format of passwords, what is the maximum time that it would take to break into an account?

(ii) In a bid to improve security, IT services propose to either double the number of letters available (by including lower case letters) or double the length (from three to six). Which of these options would you recommend? Is there a fundamental principle here that could be applied in other situations?

(iii) Suppose, to be on the safe side, IT services double the number of letters, include numbers and increase the password length to twelve. You have forgotten your password. You remember that it contains the characters $\{t, t, t, S, s, s, I, i, i, c, a, 3\}$. If you can check passwords at the same rate as a hacker, how long will it take you to get into your account?

(b) Let Z be a random variable with density

$$f_Z(z) = \frac{1}{2}e^{-|z|}, \qquad \text{for } 0 < z < \infty.$$

(i) Show that f_Z is a valid density.

(ii) Find the moment generating function of Z and specify the interval where the MGF is well-defined.

(iii) By considering the cumulant generating function or otherwise, evaluate $E(Z)$ and $Var(Z)$.

Question 3 **[20 marks, 4+6+6+4]**

(a) In a special promotion, a garage issues a token for every $SZL10$ worth of petrol purchased. Each token bears one of 6 symbols, with equal likelihood, and any customer who acquires a complete set of the 6 symbols wins a prize. Find the probability that a customer who acquires 12 tokens on visits to the garage will win a prize.

(b) Consider the following game involving two players and a bag containing 3 disks; 2 blue and 1 red. The players take turns. At each turn the player puts \$$X$ into the kitty, removes a disk from the bag, looks at the colour and replaces it in the bag. If the disk is blue, the game continues with the other players turn. If the disk is red the game stops and the player who picked the red disk wins the money in the kitty. Suppose $X \sim Exp(\theta)$. Let Y be the number of turns in a game ($Y = 1$ if the red disk is chosen on the first turn). Let Z be the amount of money in the kitty when the game ends.

(i) Evaluate $P(Y = 1)$, $P(Y = 2)$ and $P(Y = 3)$. Write down the probability mass function of Y.

(ii) Derive the moment generating function of X and the moment generating function of Y. For each give an interval around the origin for which the function is well defined.

(iii) Evaluate the probability that the person who starts the game wins the game. Given the choice, would you choose to start or go second in the game? Give reasons.

Question 4 [20 marks, 6+6+8]

(a) The annual number of hurricanes forming in the Atlantic basin has a Poisson distribution with parameter λ. Each hurricane that forms has probability p of making landfall independent of all other hurricanes. Let X be the number of hurricanes that form in the basin and Y be the number that make landfall. Find:

(i) $E(Y)$,

(ii) $\text{Corr}(X, Y)$.

(b) Let X be such that the distribution of X given $Y = y$ is Poisson, parameter y. Let $Y \sim \text{Poisson}(\mu)$. Show that

$$G_{X+Y}(s) = \exp\left\{\mu(se^{s-1} - 1)\right\}$$

Question 5 [20 marks,10+10]

(a) An insurance company offers annual motor-car insurance based on a "no claims discount" system with levels of discount 0%, 30% and 60%. A policyholder who makes no claims during the year either moves to the next higher level of discount or remains at the top level. If there is exactly one claim during the year, the policyholder either moves down one level or stays at the bottom level (0%). If there is more than one claim during the year, the policyholder either moves down to or stays at the bottom level. For a particular policyholder, it may be assumed that claims arise in a Poisson process at rate $\lambda > 0$. Explain why the situation described above is suitable for modelling in terms of a Markov chain with three states, and write down the transition probability matrix in terms of λ.

(b) Suppose that the genders of all children in a family are independent and that boys and girls are equally probable, that is, both have probability 0.5. Let the probability p_n that a family has exactly n children be $(1-p)p^n$ for $n = 0, 1, \cdots$ with p such that $0 < p < 1$. Show that the probability that a family contains exactly k boys, is given by

$$2(1-p)p^k(2-p)^{-(k+1)}.$$

Question 6 [20 marks,6+6+8]

(a) If the joint density function of X and Y is

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

find the marginal density for

(i) $U = Y - X$,

(ii) $V = X + Y$.

(b) A homogeneous Markov chain $\{X_n \ : \ n = 0, 1, \cdots\}$ has states $\{0, 1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

At time $n = 0$, the system is equally likely to be in any of the states $0, 1, 2$. Find $P(X_2 = 1)$ and $P(X_2 = 2)$.