UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2012

TITLE OF PAPER	:	PROBABILITY THEORY
COURSE CODE	:	ST 201
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	ANSWER ANY FIVE QUESTIONS.
REQUIREMENTS	:	SCIENTIFIC CALCULATOR AND STATISTICAL TABLES.

Question 1

A Personal Identification Number (PIN) consists of four digits in order, each of which may be any one of 0, 1, 2, ..., 9.

a) Find the number of PINs satisfying each of the following requirements.
   (i) All four digits are different.
   (ii) There are exactly three different digits.
   (iii) There are two different digits, each of which occurs twice.
   (iv) There are exactly three digits the same.

(9 Marks)

b) Two PINs are chosen independently and at random, and you are given that each PIN consists of four different digits. Let X be the random variable denoting the number of digits that the two PINs have in common.

   (i) Explain clearly why $P(X = k) = \dfrac{\binom{4}{k}\binom{6}{4-k}}{\binom{10}{4}}$, for $k = 0, 1, 2, 3, 4$.

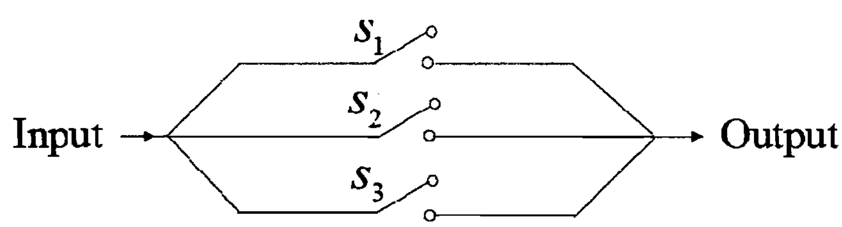
(4 Marks)

   (ii) Hence write down the values of the probability mass function of X, and find its mean and variance.

(7 Marks)

Question 2

Three switches connected in parallel operate independently. Each switch remains closed with probability p.



(a) Find the probability of receiving an input signal at the output.

(10 Marks)

(b) Find the probability that switch S_1 is open given that an input signal is received at the output.

(10 Marks)

Question 3

(a) The random variable X has the binomial distribution with probability mass function

$$P(X=x)=\binom{2}{x}p^x(1-p)^{2-x}, \quad x=0, 1, 2; \quad 0<p<1.$$

Write down E(X), Var(X) and $P(X = 2)$ in terms of the parameter p. Also find $P(X = 0 \mid X < 2)$ and $P(X = 1 \mid X < 2)$, simplifying your answers as far as possible.

(12 Marks)

(b) The random variable T follows the exponential distribution with rate parameter λ, with the probability density function (pdf) of T given by

$$f_T(t)=\lambda e^{-\lambda t}, \quad t>0, \quad \lambda>0.$$

Obtain the cumulative distribution function (cdf) $F_T(t)$ of T, and Show that $P(a < T \leq b) = e^{-\lambda a} - e^{-\lambda b}$.

(8 Marks)

Question 4

Flaws in lengths of fibre optic cable made by Company A occur in a Poisson process at rate λ_A per metre length, so that the number of flaws X in a length of l metre of rope has the Poisson probability mass function

$$P(X=x)=\frac{\exp(-\lambda_A l).(\lambda_A l)^x}{x!}, \quad x=0, 1, 2, \ldots; \ \lambda_A>0.$$

(a) Find the probability that there are (i) no flaws, (ii) more than 2 flaws, in a 1000-metre length of rope made by company A, given that $\lambda_A = 0.002$.

(4 Marks)

(b) Company B makes similar cable, indistinguishable in appearance from that made by Company A, in which flaws occur in a Poisson process at rate $\lambda_B = 0.003$ per metre. A communications system is installed with 100 metres of rope from Company A and 100 metres of rope from Company B. Assuming that the lengths of cable supplied by A and B are independent, find the probability that (i) there are no flaws, (ii) there is exactly one flaw, in the installation.

(5 Marks)

(c) A telecommunications company buys 75% of cables from Company A and 25% from Company B. The supplier's label has become detached from a drum of cable of length 2 km which is found to have 7 flaws. Find the probability that this drum was supplied by Company A.

(6 Marks)

(d) Suppose, instead, that the cable in this drum had been found to have 8 flaws. Find the probability that this drum was supplied by Company A. Compare this probability with your answer to part (c) and comment.

(5 Marks)

Question 5

The joint probability density function of the random variables X and Y is

$$f(x, y) = \frac{1}{2\pi}\exp\left(-\tfrac{1}{4}(x-1)^2 - \left(y - \tfrac{1}{4}(1+x)\right)^2\right), \quad -\infty < x < \infty, \quad -\infty < y < \infty.$$

(a) Show that X has the Normal distribution with mean 1 and variance 2.

(7 Marks)

(b) Show that the moment generating function of X is $m_X(t) = \exp(t + t^2)$

(7 Marks)

(c) Use the moment generating function to find $E(X^3)$.

(6 Marks)

Question 6

Two tennis players, A and B, are playing a match. Let X be the number of serves faster than 125 km/h served by A in one of his service games and let Y be the number of these serves returned by B. The following probability model is proposed:

$P(X = 0) = 0.4$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$ and $P(X = 3) = 0.1$.

The conditional distribution of Y (given that $X = x > 0$) is binomial with parameters x and 0.4, and $P(Y = 0 \mid X = 0) = 1$. Assume that this model is correct when answering the following questions.

(a) Find the joint probability distribution of X and Y and display it in the form of a two-way table.

(7 Marks)

(b) Find the marginal distribution of Y and evaluate E(Y).

(4 Marks)

(c) Find Cov(X, Y).

(4 Marks)

(d) Use your joint probability distribution table to find the probability distribution of the number of serves faster than 125 km/h that are not returned by B in a game.

(5 Marks)

Question 7

Suppose that X and Y are independent random variables with the same probability density function (pdf) f (x).

(a) Write down, without proof, a formula for the pdf of $X + Y$.

(2 Marks)

(b) Suppose that $f(x) = x/2$ for $0 < x < 2$ (and $f(x) = 0$ elsewhere). Find the pdf of $W = X + Y$ for $0 < w < 2$ and for $2 < w < 4$.

(12 Marks)

(c) Find the pdf of $V = (X - 1)^2$.

(6 Marks)