UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2011

TITLE OF PAPER	:	MATHEMATICS FOR STATISTICIANS
COURSE CODE	:	ST 202
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	THIS PAPER HAS SIX (6). ANSWER ANY THREE (3) QUESTIONS.

Question 1 **[20 marks, 8+12]**

Let

$$C = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

(a) Find the eigenvectors of C and for each eigenvalue find the eigenvectors.

(b) Diagonalise C; that is, find complex matrices P (invertible) and D (diagonal) such that $P^{-1}CP = D$. Show that the two eigenvectors you found for C are orthogonal.

Question 2 **[20 marks, 2+4+8+6]**

(a) What does it mean to say that a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is linearly independent.

(b) Show that the vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$$

are linearly dependent. Express $\boldsymbol{v}_3$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

(c) Starting with vector $\boldsymbol{v}_1$ find an orthonormal basis of $\mathbb{R}^3$.

(d) Use L'Hôpital's Rule to determine the following limit:

$$\lim_{x \to +\infty} \frac{\ln x}{x}$$

Question 3 **[20 marks, 4+6+5+5]**

(a) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation given by

$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + y + z + w \\ y - 2z + 2w \\ x + 3z + w \end{pmatrix}.$$

(i) Write down a matrix A_T which represents T with respect to the standard basis in $\mathbb{R}^4$ and $\mathbb{R}^3$; that is, such that $T(\boldsymbol{x}) = A_T \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^4$.

(ii) Find all vectors $\boldsymbol{x}$ for which $T(\boldsymbol{x}) = \boldsymbol{b}$ where

$$\boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

(b) Consider the function

$$f(x, y, z) = \ln(xy + z)$$

(i) Find the gradient of f at the point (x, y, z).

(ii) Find the normal vector and the tangent plabe at the point $(1, 1, 0)$ to the surface $\ln(xy+z) = 0$.

Question 4 [20 marks, 8+6+6]

(a) It costs a company

$$C(x, y, z) = 3x^2 + 2y^2 + z^2 + 3xy + 4xz,$$

to produce quantities X, Y and Z respectively. The company is currently producing 2, 3 and 4 units of X, Y and Z respectively and they want to reduce production so as to reduce their costs. Find the ratio in which they should reduce production of the three goods in order to make the greatest savings.

(b) Determine whether the integral

$$\int_0^1 \frac{dx}{x^{1/4} - x^{5/4}},$$

converges.

(c) Find the following limit:

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

Question 5 [20 marks, 4+3+6+7]

(a) Determine whether the integral

$$\int_1^\infty \frac{x^3 + 2}{x^6 + 3x^2},$$

converges.

(b) Consider

$$A = \begin{pmatrix} 5 & -8 & -4 \\ 3 & -5 & -3 \\ -1 & 2 & 2 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(i) Find the rank of A.

(ii) Find a general solution of the system of equation $A\boldsymbol{x} = \mathbf{0}$.

(iii) Show that $\boldsymbol{v}$ is an eigenvector of A and find the corresponding eigenvalue.

Question 6 [20 marks, 4+5+6+5]

(a) Find the derivatives $\frac{dy}{dx}$ of the following functions

(i) $x^3 + y^3 + 3xy = 3$,

(ii) $y = x^{\frac{1}{x}}$

(b) Use Newton's method with five (5) iterations and five decimal places to find the root of

$$f(x) = x^3 - x + 1$$

given that $x_0 = -1$.

(c) Evaluate the following integral, $\int_0^2 \int_{-1}^1 (1 - 6x^2 y \; dx \; dy$.