UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2012

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 6+8+6]

(a) Consider the following suggested model for the lifetime of an electrical component. Each component has a quality factor q; and the distribution of the lifetime of the component has pdf

$$f(x) = \begin{cases} qe^{-qx} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $q > 0$. The value of q varies between components. The quality factor of a randomly chosen component has pdf

$$f(q) = \begin{cases} \beta e^{-\beta q} & q > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\beta > 0$ is a parameter. Find the pdf of the lifetime of a randomly chosen component.

(b) X has pdf

$$f(x) = \begin{cases} \frac{kx^3}{(1+2x)^6} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $Y = \frac{2X}{1+2X}$ and identify the constant k.

(c) X_1, X_2 and X_3 are independent, each uniformly distributed on the interval $(0, 1)$. Find $\mathbb{P}(X_1 < X_2 < X_3)$.

Question 2 [20 marks, 3+4+3+6+4]

(a) A regular insurance claimant is trying to hide 3 fraudulent claims among 7 genuine claims. The claimant knows that the insurance company processes claims in batches of 5 or in batches of 10. For batches of 5, the insurance company will investigate one claim at random to check for fraud; for batches of 10, two of the claims are randomly selected for investigation. The claimant has three possible strategies:

(i) submit all 10 claims in a single batch,

(ii) submit two batches of 5, one containing 2 fraudulent claims the other containing 1,

(iii) submit two batches of 5, one containing 3 fraudulent claims the other containing 0.

What is the probability that all three fraudulent claims will go undetected in each case? What is the best strategy?

(b) Let Y be a random variable that has a Bernoulli distribution with parameter p, that is, $P(Y = 1) = p$ and $P(Y = 0) = 1 - p$. **Without using generating functions**:

(i) Show that $E(Y^r) = p$ for $r = 1, 2, \cdots$.

(ii) Find the third central moment of Y.

Question 3 [20 marks, 6+6+8]

(a) If both X and Y are two independent Geometric random variables with parameter p, given $X+Y=z$, prove that the conditional distribution of X is uniform with parameter $z-1$.

(b) If $Z \sim$ Polya(r,p), then the mass function of Z is

$$f_Z(z) = \begin{cases} \frac{\Gamma(r+z)}{z!\Gamma(r)} p^r (1-p)^z & \text{for } z = 0, 1, 2, 3, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the **Polya** distribution is a valid distribution.

(c) Use the moment generating function and cumulant generating function for a negative binomial to find the first and second central moments of a negative binomial distribution.

Question 4 [20 marks, 2+4+6+8]

Consider the function

$$f_{X,Y}(x,y) = \begin{cases} x+y & \text{for } 0 < x, y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $f_{X,Y}$ is a valid density.

(b) Derive the joint distribution function of X and Y.

(c) Find the marginal density of X.

(d) Find the correlation of X and Y.

Question 5 [20 marks, 6+8+6]

(a) Bob regularly frequents a certain cake shop and on each visit Bob buys buns and doughnuts. Let B and D denote the total number of buns and doughnuts that Bob buys on one particular visit. The (B,D) have the joint probability mass function.

	D		
B	1	2	4
0	k^2	$2k$	$k(1-k)$
1	k	k^2	k
2	$k(1-k)$	$2k$	0

Calculate the covariance of B and D.

(b) If both X and Y are two independent Geometric random variables with parameter p, prove that $X+Y$ is a Pascal (Negative Binomial) random variable with parameters 2 and p.

(c) Let X and Y be independent random variables with $X \sim$ Gamma(α_1, λ_1) and $Y \sim$ Gamma(α_2, λ_2). Give an expression for the moment generating function of Z where $Z = X+Y$. Under what conditions does Z have a Gamma distribution?