UNIVERSITY OF SWAZILAND

MAIN EXAMINATION PAPER 2012

TITLE OF PAPER	:	MATHEMATICS FOR STATISTICIANS
COURSE CODE	:	ST 202
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	THIS PAPER HAS SIX (6). ANSWER ANY THREE (3) QUESTIONS.

Question 1 **[20 marks, 10+10]**

(a) For some numbers a, b, c, the function f is given by

$$f(x) = ax^2 + bx^3 + cx^4.$$

Given that

$$f(1) = 15, \qquad f'(1) = 44 \quad , \text{ and } \int_0^1 f(x)dx = 4,$$

find the numbers a, b and c.

(b) Determine each of the following integrals

$$\int e^{\sqrt{x}}dx,$$

$$\int \frac{\cos x}{(1-\sin x)(\sin x + 2)}dx.$$

Question 2 **[20 marks, 8+12]**

Consider

$$A = \begin{pmatrix} 5 & -8 & -4 \\ 3 & -5 & -3 \\ -1 & 2 & 2 \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(a) Find the rank of the matrix A. Find the general solution of the system of equations $A\boldsymbol{x} = 0$.

(b) Show that the vector $\boldsymbol{v}$ is an eigenvector of A and find the corresponding eigenvalue. Find all the eigenvectors of A which correspond to this eigenvalue. Hence find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Question 3 **[20 marks, 8+6+6]**

(a) Find the following limits:

$$\lim_{x\to 0} \frac{xe^x - (e^x - 1)}{x^2}, \qquad \lim_{x\to 0} \frac{1}{x} \ln\left(\frac{e^x - 1}{x}\right).$$

(b) Let A and B be $n \times n$ orthogonal matrices. Prove that $A(A^T + B^T)B = A + B$. Further show that, if $\det A + \det B = 0$, then $A + B$ is not invertible.

(c) Let A and B be 3×3 matrices, with $\det(A) = 4$ and $\det(A^2B^{-1}) = -8$. Use properties of determinants to compute $\det(2A)$ and $\det(B)$.

Question 4 [20 marks, 6+6+8]

(a) Determine the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$.

(b) An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by,

$$P(x) = -8x^2 + 3200x - 80,000$$

How many apartments should they rent in order to maximize their profit?

(c) Choose h and k such that the system below

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ 3x_1 + hx_2 &= k, \end{aligned}$$

has

(i) no solution,

(ii) a unique solution, and

(iii) many solutions.

Question 5 [20 marks, 2+6+12]

(a) Consider a Markov chain given by the system of equations, $\boldsymbol{x}_t = A\boldsymbol{x}_{t-1}$ with

$$A = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}, \qquad \boldsymbol{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \qquad t \in \mathbb{N}.$$

where x_t represents the number of workers on the day shift of a certain factory in month t, and y_t represents the number on the night shift in month t, in a total fixed population of 4000 factory workers. The owners are encouraging workers to change to the night shift from the day shift. Initially, there are only 1000 workers on the night shift.

(i) Each month, what percentage of day workers change to the night shift?.

(ii) Find the "long term" population distribution of this system. State clearly the eventual number of workers on each shift.

(b) A system of linear equation $B\boldsymbol{x} = \boldsymbol{d}$ if known to have the following general solution:

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad s, t \in \mathbb{R}.$$

Let $\boldsymbol{c}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$ be the first column of B. Find the matrix B and find the vector $\boldsymbol{d}$.

Question 6 [20 marks, 4+5+6+5]

(a) Functions f and g are as follows

$$f(x) = x^4 + 2x^3 + 2x^2 + 2, \qquad g(x) = -x^4 + 2x^3 + 18x + 20.$$

Show that the curves $y = f(x)$ and $y = g(x)$ intersect for exactly two values of x. Find these values of x. (Do not attempt to sketch the curves.)

(b) Use Newton's method with five (5) iterations and five decimal places to find the root of

$$f(x) = x^3 - x + 1$$

given that $x_0 = -1$.

(c) Let $f(x) = \frac{x^3}{5000}(10 - x)$ for $x \leq x \leq 10$ and $f(x) = 0$ for all other values of x. Show that $f(x)$ is a probability density function.