UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2013

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 **[20 marks, 3+3+6+2+6]**

Assume $\{X_n\}_{n=0}^{\infty}$ is a Markov chain (MC) with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 0.3 & 0.2 & 0.5 \\ 0.3 & 0 & 0.5 & 0.2 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}.$$

(a) Find the two step transition probability matrix.

(b) Suppose that the probability function of X_1 is given by the vector $\beta = (0, 0.5, 0, 0.5)$. Find the probability function of X_3.

(c) Classify the state space. For each class, determine whether it is recurrent or transient. Determine their periods.

(d) What does it mean by "irreducible"? Is this MC reducible?

(e) Find the long run proportions of times when the MC is in state 0, in state 2. (Do not blindly solve $\pi\mathbf{P} = \pi$).

Question 2 **[20 marks, 8+6+6]**

(a) Suppose $N \sim$ Geometric(p) and each $X_i \sim$ Binomial(1,θ) (independent). Find the distribution of $Z = \sum_{i=1}^{N} x_i$.

(b) Suppose that X and Y are continuous random variables with joint pdf given by

$$f)X,Y(x,y) = \frac{1}{2x^2y}, \qquad 1 \le x < \infty, \frac{1}{x} \le y < x,$$

and zero otherwise.

Derive

(i) the **marginal** pdf of Y,

(ii) the **conditional** pdf of X given $Y = y$.

Question 3 **[20 marks, 8+8+1+1+2]**

(a) Suppose that $X$is a continuous random variable with pdf

$$f_X(x) = \exp\{-(x+2)\}, \qquad -2 < x < \infty.$$

Find the mgf of X, and **hence** find the expectation and variance of X.

(b) Consider a single-server queueing system in which the service time is negative exponential with mean μ^{-1} and customer arrivals form a Poisson process with rate λ, except that any customer arriving when there are already N customers in the system leaves without joining the queue. Show that the steady-state distribution of the number of customers in the system is

$$\pi_n = \rho^n(1-\rho)(1-\rho^{N+1})^{-1}, \qquad 0 \le n \le N,$$

where $\rho = \lambda/\mu$.

(c) Suppose that $E(X) = 3$, $E(Y) = 2$, $Var(X) = 5$, $Var(Y) = 4$, and $Cov(X, Y) = -2$.

(i) Find $E(2X + 3Y)$.

(ii) Find $Var(2X - 3Y)$.

(iii) Find the correlation between X and Y.

Question 4 **[20 marks, 8+8+4]**

(a) A homogeneous Markov chain $\{X_n : n = 0, 1, \cdots\}$ has states $\{0, 1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Determine the limiting distribution.

(b) A continuous random variable X has cdf given by

$$F_X(x) = \frac{2\beta x}{\beta^2 + x^2}, \qquad 0 \le x \le \beta,$$

for some constant $\beta > 0$. Find the pdf of X, and show that the expectation of X is

$$\beta(1 - \log 2).$$

(c) In an accelerated life experiment, the times to failure, in hours, of a certain type of device have probability density function

$$f(x) = \nu^2 x e^{-\nu x}$$

for $x > 0$. Show that the mean time to failure is $2/\nu$.

Question 5 **[20 marks,10+4+6]**

(a) Let $N \sim$ Poisson(μ). Define the random variable

$$Y = \begin{cases} X_1 \sim \text{Exponential}(\lambda) & N > 0; \\ X_2 \sim \text{Exponential}(2\lambda) & N = 0, \end{cases}$$

where N, X_1 and X_2 are independent of each other. Derive the moment generating function of Y, and find $E(Y)$ and Var(Y).

(b) Let X and Y be two independent standard normal random variables. Consider random variables U and V such that X and Y can be represented by

$$\begin{cases} X = U \cos V, \\ Y = U \sin V. \end{cases}$$

You are given some useful properties of sin and cos functions:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \sin^2(x) + \cos^2(x) = 1.$$

Also, over $[0, 2\pi)$, $\sin \ge 0$ for $x \in [0, \pi]$ and $\cos x \ge 0$ for $xin[0, \pi/2]$ of $[3\pi/2, 2\pi)$,

(i) Give the respective ranges for U and V in order that the transformation defined is one to one. With this, find U and V in terms of X and Y.

(ii) Find the joint probability density of $f_{U,V}(u,v)$ of U and V.

Question 6 [20 marks, 8+3+4+5]

(a) We agree to try to meet between 12 and 1 for lunch at our favorite sandwich shop. Because of our busy schedules, neither of us is sure when we'll arrive; we assume that for each of us our arrival time is uniformly distributed over the hour. So that neither of us has to wait too long, we agree that we will each wait exactly 15 minutes for the other to arrive, and then leave.

What is the probability we actually meet each other for lunch?

(b) Without a vaccine, the death rate is 0.1 for a patient with disease D_1, and 0.5 for a patient with disease D_2. If a patient receives vaccine A and still develops a disease, the death rate is 0.06 for disease D_1 and 0.1 for disease D_2. If a patient receives vaccine B and still develops a disease, the corresponding death rates are 0.01 and 0.2, respectively.

(i) Show that for any events A, B and C, we have

$$P(A \cap B \cap C) = P(C|A \cap B)P(B|A)P(A)$$

(ii) Find the probability that a particular individual satisfies the conditions that he/she is not vaccinated, develops disease D_2, and dies eventually.

(iii) Given that a patient developed disease D_2 and died, find the probability that the patient has not been vaccinated.