UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2013

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 4+8+8]

(a) The joint probability density function of (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} k(x+y), & 0 < x < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find k.

(ii) Find $f_X(x)$ and $f_Y(y)$, the marginal density function of X and Y respectively. Are X and Y independent?

(b) Let X be such that the distribution of X given $Y = y$ is Poisson, parameter y. Let $Y \sim$ Poisson(μ). Show that the probability generating function of $X + Y$ is

$$G_{X+Y}(s) = \exp\left\{\mu(se^{s-1} - 1)\right\}.$$

Question 2 [20 marks, 8+3+3+4+2]

(a) We are playing a tournament in which we stop as soon as one of us wins n games. We are evenly matched, so that each of us wins any game with probability 1/2 independently of other games. What is the probability that the loser has won exactly k games when the match is over?

(b) The time to recovery T of a disease follows an exponential distribution with mean 3 months, i.e. $T \sim$ Exp(1/3).

(i) Derive the standard deviation of time to recovery.

(ii) Show that the probability that recovery takes longer than a year is e^{-4}.

(iii) There are 10 patients with the disease in a ward with recovery times independent of each other. What is the probability that at least 2 of them need over a year to recover? Give your answer to 3 significant figures.

(iv) State the expected number of patients who need over a year to recover in the ward.

Question 3 [20 marks, 8+4+8]

(a) In a lecture hall containing 100 people, you consider whether or not there are three people in the room who share the same birthday. Calculate this probability.

(b) In a multiple-choice test, the probability that you know the answer to a question is 0.6. If you do not know the answer, you choose one at random. Suppose that there are 10 questions in the test and let C be the number of multiple choices per question (the test will have all the questions having the same number of choices C). Before taking the test, you do not know the value of C, but you know that it can be either 3 or 4, with probability 0.3 and 0.7 respectively.

(i) Find the probability that you know the answers to at least 80% of the questions.

(ii) Assume that $C = 3$. Find the conditional probability that you answer at least 90% of the questions correctly, given that you know the answers to at least 80% of the questions.

Question 4 [20 marks, 4+7+5+4]

In an icecream shop, the sales of the number of cones of icecream in a particular day is denoted by X. It follows a Poisson distribution with mean N. The parameter N itself is a random variable with

$$N = \begin{cases} 100, & \text{with probability } 1/3; \\ 200, & \text{with probability } 1/3; \\ 300, & \text{with probability } 1/3. \end{cases}$$

(a) Find the probability $P(X = 0)$.

(b) Find $E(X)$ and Var(X).

(c) Given that $X = x$, what is the probability that $N = 100$? Simplify your answer as much as possible.

(d) Suppose that instead of the distribution given above we have $N \sim$ Exp(λ). Find $E(X)$ as a function of λ.

Question 5 [20 marks,8+2+2+2+2+4]

(a) An insurance company offers annual motor-car insurance based on a "no claims discount" system with levels of discount 0%, 30% and 60%. A policyholder who makes no claims during the year either moves to the next higher level of discount or remains at the top level. If there is exactly one claim during the year, the policyholder either moves down one level or stays at the bottom level (0%). If there is more than one claim during the year, the policyholder either moves down to or stays at the bottom level. For a particular policyholder, it may be assumed that claims arise in a Poisson process at rate $\lambda > 0$. Explain why the situation described above is suitable for modelling in terms of a Markov chain with three states, and write down the transition probability matrix in terms of λ.

(b) A and B are events of positive probability. Supply proof of each of the following.

(i) If A and B are independent, A and B^c are independent.

(ii) If A and B are independent, $P(A^c|B^c) + P(A|B) = 1$.

(iii) If $P(A|B) < P(A)$, then $P(B|A) < P(B)$.

(iv) If $P(B|A) = P(B|A^c)$ then A and B are independent.

(c) Consider the mutually exclusive and exhaustive events A_0, A_1 and A_2. Is it possible to have $P(A_0 \cup A_1) = \frac{1}{2}$, $P(A_1 \cup A_2) = \frac{1}{2}$ and $P(A_2 \cup A_0) = \frac{2}{3}$?

Question 6 [20 marks, 8+8+4]

(a) A homogeneous Markov chain $\{X_n : n = 0, 1, \cdots\}$ has states $\{0, 1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

At time $n = 0$, the system is equally likely to be in any of the states $0, 1, 2$. Find $P(X_2 = 1)$ and $P(X_2 = 2)$.

(b) You flip a coin which shows a head with probability p and a tail with probability $q = 1 - p$. If the first flip shows a head, you record the number of consecutive heads, X, you get from the second flip onwards. Then you record the number of consecutive tails, Y , you get after the consecutive run of heads is over. Similarly, if the first flip shows a tail, X denotes the consecutive number of tails from the second flip onwards, and Y denotes the number of consecutive heads after the run of consecutive tails is over. Show that the joint probability mass function of X and Y is

$$p_{X,Y}(x, y) = p^{x+2}q^y + q^{x+2}p^y, \qquad x, y \text{ are integers}, \quad x \geq 0, y \geq 1.$$

(c) Starting from the definition of a probability measure, show that $P((A \cup B) \cap (A^c \cup B^c)) = P(A) + P(B) - 2P(A \cap B)$.