UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2014

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 **[20 marks, 5+7+8]**

Suppose X and Y are independent random variables having Poisson distributions with respective means $\lambda(>0)$ and $\mu(>0)$.

(a) Show that $X+Y$ also follows a Poisson distribution.

(b) Find $\mathbb{P}(X=k|X+Y=n)$ when k and n are integers with $0 \leq k \leq n$. For given fixed $n>0$, name the distribution you have obtained.

(c) Telephone calls arriving at a computer helpline are classed as urgent or standard, urgent calls average 8 per hour, standard calls average 24 per hour. Ten calls arrive within 30 minutes, find (to two significant figures) the probability that at most two of them are urgent, stating any assumptions you make.

Question 2 **[20 marks, 2+6+12]**

A factory has two production lines, line 1 and line 2, for manufacturing car seats.

The preferred arrangement is to run both lines at standard speed. When running at standard speed, the lifetime of line 1 has an exponential distribution with a mean of 30 days and the lifetime of line 2 has an exponential distribution with a mean of 15 days. There is one repair crew to deal with line failures, and repair times are exponentially distributed with mean 2 days. If one line fails, the other will be run at double speed in order to meet production targets. In this case, the means of the lifetime distributions are then reduced to 10 days and 5 days for lines 1 and 2 respectively.

If both lines fail, the repair crew will repair line 1, because it is the more reliable, even if this means abandoning repair of line 2.

The lifetimes and repair times are statistically independent.

(a) According to this protocol, if the repair crew is repairing line 2 and line 1 fails, the crew immediately moves to line 1, abandoning the repair on line 2 until later. Explain why in setting up a model it is unnecessary to allow for the time spent initially repairing line 2.

(b) Set up a continuous time Markov chain model for the state of the factory, defining the state space and writing down the instantaneous transition rates.

(c) Find the corresponding equilibrium distribution, expressing the values as fractions and then calculating them correct to 2 decimal places. What, correct to 2 decimal places, is the long-term proportion of time during which neither line is running so that the factory is unable to meet the production target?

Question 3 **[20 marks, 7+5+8]**

The continuous random variables X and Y have joint probability density function

$$f(x,y) = \frac{1}{x}e^{-x}, \qquad 0 < y < x.$$

(a) Show that X has an exponential distribution. Hence show that, conditional on $X=x$, Y has the uniform distribution on the interval $(0,x)$.

(b) Show that, for non-negative integers m and n,

$$E(X^m Y^n) = \frac{(m+n)!}{n+1}.$$

(c) Use the result proved in part (b) to obtain $E(X)$, Var(X), $E(Y)$ and Var(Y). Find and interpret the value of the correlation between X and Y.

[You may use without proof the result that, for any non-negative integer r,

$$\int_0^\infty u^r e^{-u} du = r!]$$

Question 4 [20 marks, 4+2+4+10]

An experiment has n possible outcomes labelled $1, 2, \cdots, n$, each with probability $\frac{1}{n}$, for some $n > 1$.

(a) Suppose that $n = 8$, and $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5, 6\}$ and $C = \{1, 3, 7, 8\}$.

(i) Show that $P(A \cap B) = P(A)P(B)$, and that $P(A \cap B \cap C) = P(A)P(B)P(C)$.

(ii) By considering another intersection, show that A, B and C are not independent.

(iii) Construct an event D so that A, B and D are independent.

(b) By considering the cases $n = 6m + r$ for $r = 0, 1, 2, 3, 4, 5$, find the values of n for which the events

$$E = \{\text{Outcome divisible by 2}\} \text{ and } F = \{\text{Outcome divisible by 3}\}$$

are independent.

Question 5 [20 marks, 10+6+4]

(a) State the assumptions for a Poisson process for incidents occurring in time at rate λ, and use them to show that $p_0(t)$, the probability that there is no incident by time t, starting from time zero, is given by

$$p_0(t) = e^{-\lambda t}.$$

Deduce that the time to the first incident has an exponential distribution.

(b) Now suppose that the rate of the process, instead of being constant, varies with time t and is in fact equal to t.

(i) Show that, for this process,

$$p_0(t) = e^{-\frac{1}{2}t^2}.$$

and find the expected time to the first incident. [You may use, without proof, the result that $\int_0^\infty \sqrt{u} e^{-u} du = \frac{\sqrt{\pi}}{2}$.]

(ii) If the first incident is known to have occurred at time s, show that the probability that no incidents occur in $(s, s+t]$ is given by

$$p_0(t|s) = \exp\left(-st - \frac{1}{2}t^2\right).$$

Deduce that the independence property of the intervals between incidents, which is known to hold for the constant rate process, fails for this variable rate process.

Question 6 [20 marks, 4+3+3+3+7]

In the Ultrastratos civilisation on planet Anachronista, the population is divided into four strata which, in order of status, are labelled Alpha, Beta, Gamma and Delta. By the traditions of the civilisation, no child can have a status more than one different from its parents. As examples, in each generation 20% of the children of Alphas grow up to be Betas, the rest remaining Alphas, while of the Beta offspring 50% remain Betas while 10% become Alphas and the rest Gammas.

A Markov chain describes the status of members of the population at successive generations. Its transition matrix is given by $\mathbf{P}$, defined below, in which some entries labelled $*$ have been omitted.

$$\mathbf{P} = \begin{array}{c} \textit{Alpha} \\ \textit{Beta} \\ \textit{Gamma} \\ \textit{Delta} \end{array} \begin{pmatrix} * & 0.2 & * & 0 \\ 0.1 & 0.5 & * & * \\ * & * & 0.5 & 0.4 \\ * & 0 & * & 0.8 \end{pmatrix}$$

(a) Use the information above to fill in the missing elements of $\mathbf{P}$ and calculate the two-step-ahead transition matrix.

(b) What is the probability that the child of a Beta becomes a Gamma? Show that the grandchild of a Gamma is twice as likely to become a Delta as the grandchild of a Delta is to become a Gamma.

(c) Calculate the probability that, after four generations, a descendant of an Alpha is a Delta.

(d) Explain how you can tell that the chain consists of a single irreducible class. Are any of the states transient? Give your reason.

(e) Find the stationary distribution of the chain. If the population initially has no Deltas, what will be the proportion of Deltas after a large number of generations?