UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2015

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 10+10]

The continuous random variables X and Y have joint probability density function with $f(x, y) = kxy$ if $0 < x < y < 1$, with $f(x, y) = 0$ elsewhere, where k is a constant.

(a) Evaluate k, and find the marginal probability densities of X and Y. Say, with a reason, whether or not X and Y are independent.

(b) Show that, for all non-negative integers r and s,

$$E(X^r Y^s) = \frac{8}{(r+2)(r+s+4)}$$

Hence find the correlation between X and Y.

Question 2 [20 marks, 4+8+4+4]

Consider a Markov chain with three states, 1, 2 and 3, and transition matrix

$$\begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

(a) Explain what is meant by the statement that a Markov chain is an irreducible recurrent chain, and show, stating any general results that you assume, that this statement is true for the present chain.

(b) Find the stationary distribution for this chain.

Now consider this Markov chain as a simple model for social mobility. People's occupations have been classified into the three classes "Upper" (State 1), "Middle" (State 2) and "Lower" (State 3). The transition probabilities model how the occupational classes of sons depend on the occupational classes of fathers. The transition probabilities as given above are rounded versions of estimates obtained from a social survey.

(c) If initially, in the first generation, the proportions of males in each class are $\frac{2}{5}$, $\frac{2}{5}$ and $\frac{1}{5}$ respectively, what proportions would you expect to find in each class at the second generation?

(d) If, in a large population, this transition matrix remains unchanged over a number of generations, approximately what proportions of males would you expect to find in each of the three occupational classes after several generations?

Explain carefully your reasoning and state any results about stationary distributions that you assume.

Question 3 [20 marks, 4+6+6+4]

In a television quiz show, a contestant is asked a succession of questions. Assume that for each question, independently of the answers to any other questions, the probability of a correct answer is θ and the probability of an incorrect answer is $1-\theta$, where $0 < \theta < 1$. At the end of the game, the prize depends on the number of successive correct answers given since the last incorrect answer or, if all have been answered correctly, the number of questions. Let $\{X_n\}(n \geq 0)$ denote a Markov chain in which X_n represents the situation after the n^{th} question, namely the number of correct answers since the last incorrect answer, with $X_0 = 0$ assumed by convention. After each question the number of correct answers since the last incorrect answer either increases by one or falls to zero.

(a) Assuming first that there is no upper bound on the number of questions that will be asked, write down the state space Ω and the transition probabilities $\{p_{ij}\}$ for the Markov chain $\{X_n\}$.

(b) Show that the stationary distribution $\{\pi_i\}$ for $\{X_n\}$ is given by

$$\pi_i = (1-\theta)\theta^i \qquad (i \geq 0).$$

(c) Suppose now that the contestant is to be asked exactly six questions. By calculating the six-step transition probabilities $p_{0j}^{(6)}$, find $\mathbb{P}(X_6 = j)$ for $j = 0, 1, \cdots, 6$.

(d) The contestant receives a prize of $E50 \times 2^j$ if $X_6 = j$ for $j \geq 1$. Calculate the expected winnings when $1\theta = \frac{1}{2}$.

Question 4 **[20 marks, 8+4+8]**

The discrete random variables $X_1, X_2, \cdots, X_n$ $(n \geq 2)$ are independent and each X_i has the Poisson distribution, $\mathbb{P}(X_i = x_i) = \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}$, for $\lambda_i > 0$ $(i = 1, 2, \cdots, n)$.

(a) Show that X_i has moment generating function

$$M_i(t) = \exp\{\lambda_i(e^t - 1)\}$$

Use this moment generating function to find $E(X_i)$ and $\text{Var}(X_i)$.

(b) Using moment generating functions, show that

$$S = X_1 + X_2 + \cdots + X_n$$

has the Poisson distribution with expected value $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

(c) Let $S = s$, for some $s \geq 0$. What can you say about the possible values of $(X_1, X_2, \cdots, X_{n-1})$? For $(x_1, x_2, \cdots, x_{n-1})$ in this range, obtain the conditional probability mass function

$$p_{12\cdots(n-1)|S}(x_1, x_2, \cdots, x_{n-1}|S = s)$$

.

Express this function in the form

$$\frac{s!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n}.$$

where $x_n = s - x_1 - \cdots - c_{n-1}$, stating explicitly the values of the parameters $p_1, \cdots, p_n$

Question 5 **[20 marks, 10+6+4]**

A sequence of Bernoulli trials is conducted, in which the probability of success is $p > 0$. Let W denote the number of trials needed to obtain the first success.

(a) Write down the distribution of W, and find its probability generating function (pgf). Hence or otherwise show that its mean and variance are $\frac{1}{p}$ and $\frac{1-p}{p^2}$ respectively.

(b) Deduce the mean and variance of the number of trials needed to obtain k successes, where $k \geq 2$.

(c) By using pgfs or otherwise, find the probability that exactly n trials are required to obtain k successes.

[You may quote standard properties of pgfs without proof.]

Question 6 [20 marks, 5+6+3+2+4]

A branching process starts with a single individual in generation zero. In each generation, each individual independently produces a number of offspring given by the random variable Z having probability generating function (pgf) $G(s)$.

(a) Prove that $G_{n+1}(s) = G_n(G(s))$, where $G_n(s)$ is the pgf of X_n, the number of individuals in the nth generation.

Consider the case where

$$G(s) = \frac{1+4s}{6-1}.$$

(b) Use the result in part (i) to show by induction that

$$G_n(s) = \frac{n-(n-5)s}{5-n(s-1)}, \qquad n = 0, 1, 2, 3, \cdots.$$

(c) Find the expected size of the nth generation.

(d) By considering the relevant probability for the nth generation, confirm that the process will eventually become extinct.

(e) Let T be the generation when the process becomes extinct. Show that the distribution of T is given by

$$P(T = n) = \frac{5}{(n+5)(n+4)}, \qquad n = 1, 2, 3, \cdots.$$