UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2016

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 6+6+4+4]

(a) Each time a machine is repaired it remains up for an exponentially distributed time with rate λ. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machines is exponential with rate μ_1, if it is a type 2 failure, then the repair time is exponential with rate μ_2. Each failure is , independently of the time it took the machines to fail, a type 1 failure with probability p and a type 2 failure with probability $1-p$. What proportion of time is the machine down due to a type 1 failure? what proportion of time is the machine down due to a type 2 failure? What proportion of time is it up?.

(b) Consider a usual branching process: let the population size of the nth generation be X_n and family size of the ith family in the nth generation be $Z_{n,i}$. Thus, $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$ and

$$P(Z_{1,1}=0)=\frac{1}{2}+a; \qquad P(Z_{1,1}=1)=\frac{1}{2}-2a; \qquad P(Z_{1,1}=3)=\frac{1}{4}+a;$$

for some a.

(i) Find probability generating function of the family size. When $a=\frac{1}{8}$, find the probability generating function of X_2.

(ii) Find range of a such that the probability of extinction is less than 1.

Question 2 [20 marks,10+4+6]

(a) Let $N \sim$ Poisson(μ). Define the random variable

$$Y = \begin{cases} X_1 \sim \text{Exponential}(\lambda) & N > 0; \\ X_2 \sim \text{Exponential}(2\lambda) & N = 0, \end{cases}$$

where N, X_1 and X_2 are independent of each other. Derive the moment generating function of Y, and find $E(Y)$ and Var(Y).

(b) Let X and Y be two independent standard normal random variables. Consider random variables U and V such that X and Y can be represented by

$$\begin{cases} X = U\cos V, \\ Y = U\sin V. \end{cases}$$

You are given some useful properties of sin and cos functions:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \sin^2(x)+\cos^2(x)=1.$$

Also, over $[0,2\pi)$, $\sin \geq 0$ for $x \in [0,\pi]$ and $\cos x \geq 0$ for $xin[0,\pi/2]$ of $[3\pi/2, 2\pi)$.

(i) Give the respective ranges for U and V in order that the transformation defined is one to one. With this, find U and V in terms of X and Y.

(ii) Find the joint probability density of $f_{U,V}(u,v)$ of U and V.

Question 3 [20 marks, 3+3+4+2+4+4]

(a) In 1995 an account on the UNISWA network came with a three letter (all uppercase roman letter) password. Suppose a malicious hacker could check one password every millisecond.

  (i) Assuming the hacker knows a username and the format of passwords, what is the maximum time that it would take to break into an account?

  (ii) In a bid to improve security, IT services propose to either double the number of letters available (by including lower case letters) or double the length (from three to six). Which of these options would you recommend? Is there a fundamental principle here that could be applied in other situations?

  (iii) Suppose, to be on the safe side, IT services double the number of letters, include numbers and increase the password length to twelve. You have forgotten your password. You remember that it contains the characters $\{t, t, t, S, s, s, I, i, i, c, a, 3\}$. If you can check passwords at the same rate as a hacker, how long will it take you to get into your account?

(b) Let $\{Z_n\}_{n=0}^{\infty}$ be a usual branching process with $Z_0 = 1$ and $Z_n = \sum_{j=1}^{Z_{n-1}} X_{n-1,j}$ for $n > 0$ with family sizes $X_{n,j}$ being iid random variables.

Assume $X_{0,1}$ has discrete uniform distribution on $0, 1, \cdots, k$ for some positive integer k. For example, if $k = 3$, then $P(X_{0,1} = j) = 0.25$ for $j = 0, 1, 2, 3$.

  (i) For what values of k the probability of extinction is 1?

  (ii) When $k = 3$, compute the probability of extinction.

  (iii) When $k = 5$, calculate the mean and variance of X_5.

Question 4 [20 marks, 4+6+5+5]

(a) In a special promotion, a garage issues a token for every $SZL10$ worth of petrol purchased. Each token bears one of 6 symbols, with equal likelihood, and any customer who acquires a complete set of the 6 symbols wins a prize. Find the probability that a customer who acquires 12 tokens on visits to the garage will win a prize.

(b) For a branching process with family size distribution given by.

$$P_0 = \frac{1}{6}, \qquad P_2 = \frac{1}{3}, \qquad P_3 = \frac{1}{2};$$

calculate the probability generating function of Z_2 given $Z_0 = 1$, where Z_2 is the population of the second generation. Also find, the mean and variance of Z_2 and the probability of extinction.

(c) The manager of a market can hire either Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of \$C per hour. The manager estimates that, on the average, each customer's time is worth \$1 per hour and should be accounted for the model. If customers arrive at a Poisson rate of 10 per hour, then

  (i) what is the average cost per hour if Mary is hired? if Alice is hired?

  (ii) Find C if the average cost per hour is the same for Mary and Alice.

Question 5 **[20 marks, 3+9+8]**

Let X and Y be random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} ke^{-\lambda x}, & 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find k.

(b) Derive the marginal density for Y and hence evaluate $E(Y)$, $E(Y^2)$ and $Var(Y)$.

(c) Derive the conditional density, $f_{X|Y}(x|y)$, and the conditional expectation, $E[X|Y]$. Hence or otherwise, evaluate $E(X)$ and $Cov(X, Y)$.

Question 6 **[20 marks, 4+5+2+5+4]**

(a) The number of claims received at an insurance company during a week is a random variable with mean 20 and variance 120. The amount paid in each claim is a random variable with mean 350 and variance 10000. Assume that the amounts of different claims are independent.

- (i) Suppose this company received exactly 3 claims in a particular week. The amount of each claim is still random as already specified. What are the mean and variance of the total amount paid to these 3 claims in this week?
- (ii) Assume that in one week, all claims received the same payment of 300. What is the mean and variance of the total amount paid in this week?

(b) Let Z be a random variable with density

$$f_Z(z) = \frac{1}{2}e^{-|z|}, \qquad \text{for } 0 < z < \infty.$$

- (i) Show that f_Z is a valid density.
- (ii) Find the moment generating function of Z and specify the interval where the MGF is well-defined.
- (iii) By considering the cumulant generating function or otherwise, evaluate $E(Z)$ and $Var(Z)$.