UNIVERSITY OF SWAZILAND

MAIN EXAMINATION PAPER 2017

TITLE OF PAPER : PROBABILITY THEORY

COURSE CODE : ST 201

TIME ALLOWED : 3 HOURS

INSTRUCTIONS : ANSWER ANY FIVE QUESTIONS.

REQUIREMENTS : SCIENTIFIC CALCULATOR

Question 1

a) If $P(A) = 0.25$ and $P(B) = 0.8$, the show that $0.05 \leq P(A \cap B) \leq 0.25$.

(5 Marks)

b) Let A and B be Events in a sample space Ω such that $P(A) = \frac{1}{2} = P(B)$ and $P(A^c \cap B^c) = \frac{1}{3}$. Find $P(A \cup B^c)$.

(5 Marks)

c) A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective?

(5 Marks)

d) Suppose box A contains 4 red and 5 blue chips and box B contains 6 red and 3 blue chips. A chip is chosen at random from the box A and placed in box B. Finally, a chip is chosen at random from among those now in box B. What is the probability a blue chip was transferred from box A to box B given that the chip chosen from box B is red?

(5 Marks)

Question 2

A continuous random variable X has a cumulative distribution function:

$$F_X(x) = \begin{cases} 0, if\ x \leq 0 \\ \sqrt{x}, if\ 0 < x \leq 1 \\ 1, if\ x > 1 \end{cases}$$

a) Find the probability density function of X.

(4 Marks)

b) Calculate the expectation and variance of X.

(12 Marks)

c) Calculate the lower quartile of X.

(4 Marks)

Question 3

a) The random variable X is uniformly distributed on the interval (0, 1). Derive the PDF of the random variable $Y = -\ln X$.

(10 Marks)

b) Consider two independent random variables X_1 and X_2, distributed exponentially with $\lambda = 1$. That is,

$$f_X(x) = \begin{cases} e^{-x}, x \geq 0 \\ 0, otherwise \end{cases}$$

Calculate the PDF of $X_1 + X_2$.

(10 Marks)

Question 4

If the joint moment generating function of the random variable X and Y is

$$M(s,t) = \exp(s + 3t + 2s^2 + 18t^2 + 12st)$$

What is the Covariance of X and Y?

(20 Marks)

Question 5

a) Let X and Y be random variables such that X has density function

$$f_X(x) = 24x^2 \quad , \qquad 0 < x < \frac{1}{2}$$

and the conditional density of Y given X = x is

$$p(y|x) = \frac{y}{2x^2} \ , \qquad 0 < y < 2x$$

What is the conditional density of X given Y = y over the appropriate domain?

(10 Marks)

b) Let the joint density of two random variables x and y be given by

$$f(x,y) = \frac{1}{6}(x + 4y)\,, \quad 0 < x < 2, 0 < y < 1$$

Find the probability of $X \leq 1$ given that $y = \frac{1}{2}$.

(10 Marks)

Question 6

a) Let X and Y be discrete random variables with joint density

$$p(x,y) = \frac{x + 2y}{18} \ , x = 1,2; y = 1,2$$

What is the covariance σ_{XY} between X and Y.

(15 Marks)

b) If $Var(X + Y) = 3$, $Var(X - Y) = 1$, $E(X) = 1$, and $E(Y) = 2$, the what is $E(XY)$?

(5 Marks)

Question 7

a) Let X and Y be discrete random variables with joint probability mass function

$$p(x,y) = \frac{1}{21}(x + y), \qquad x = 1,2,3; y = 1,2$$

What is the conditional mean of X given Y= y, that is $E(X|y)$?

(10 Marks)

b) Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = e^{-y}, \quad 0 < x < y < \infty$$

What is the conditional variance of Y *given that* $X = x$?

(10 Marks)

Question 8

a) Let each of the independent random variables X and Y have the density function

$$f(x) = e^{-x}, \quad 0 < x < \infty$$

What is the joint density of U = X and V = 2X + 3Y and the domain on which this density is positive?

(10 Marks)

b) Let X and Y be independent random variables, each with density function

$$f(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty$$

where $\lambda > 0$. Let U = X + 2Y and V = 2X + Y . What is the joint density of U and V ?

(10 Marks)