UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2017

TITLE OF PAPER : PROBABILITY THEORY

COURSE CODE : ST 201

TIME ALLOWED : 3 HOURS

INSTRUCTIONS : ANSWER ANY FIVE QUESTIONS.

REQUIREMENTS : SCIENTIFIC CALCULATOR

Question 1

A Personal Identification Number (PIN) consists of four digits in order, each of which may be any one of 0, 1, 2, ..., 9.

a) Find the number of PINs satisfying each of the following requirements.
   (i) All four digits are different.
   (ii) There are exactly three different digits.
   (iii) There are two different digits, each of which occurs twice.
   (iv) There are exactly three digits the same.

(9 Marks)

b) Two PINs are chosen independently and at random, and you are given that each PIN consists of four different digits. Let X be the random variable denoting the number of digits that the two PINs have in common.

   (i) Explain clearly why $P(X = k) = \dfrac{\binom{4}{k}\binom{6}{4-k}}{\binom{10}{4}}$, for $k = 0, 1, 2, 3, 4$.

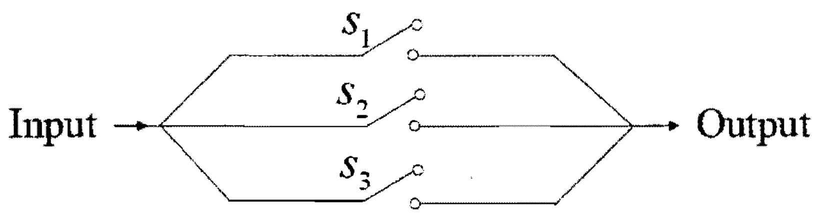
(4 Marks)

   (ii) Hence write down the values of the probability mass function of X, and find its mean and variance.

(7 Marks)

Question 2

Three switches connected in parallel operate independently. Each switch remains closed with probability p.



(a) Find the probability of receiving an input signal at the output.

(10 Marks)

(b) Find the probability that switch S_1 is open given that an input signal is received at the output.

(10 Marks)

Question 3

(a) The random variable X has the binomial distribution with probability mass function

$$P(X = x) = \binom{2}{x} p^x (1-p)^{2-x}, \quad x = 0, 1, 2; \quad 0 < p < 1.$$

Write down E(X), Var(X) and P(X = 2) in terms of the parameter p. Also find $P(X = 0 \mid X < 2)$ and $P(X = 1 \mid X < 2)$, simplifying your answers as far as possible.

(12 Marks)

(b) The random variable T follows the exponential distribution with rate parameter λ, with the probability density function (pdf) of T given by

$$f_T(t) = \lambda e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

Obtain the cumulative distribution function (cdf) $F_T(t)$ of T, and Show that $P(a < T \leq b) = e^{-\lambda a} - e^{-\lambda b}$.

(8 Marks)

Question 4

a) The random variables $X_1, \ldots\ldots, X_4$ have a joint p.d.f. given by

$$f(x_1, x_2, x_3, x_4) = \frac{3}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2), \quad 0 < x_i < 1, i = 1,2,3,4$$

i) Use the joint probability density function to compute $P(X_1 < 1/2, X_2 < 3/4, X_3 < 1, X_4 > 1/2)$

(8 Marks)

ii) Compute the marginal probability density function of (X_1, X_2)

(7 Marks)

b) If X and Y have the joint p.d.f. $f(x, y) = ce^{-x} e^{-2y}$ for $0 < x < 1$ and $x < y < 1$, what the value of the constant is c?

(5 Marks)

Question 5

a) Let the joint density of two random variables X and Y be given by

$$f(x, y) = (0.6x + 0.6 \times 4y), \quad 0 < x < 2, \qquad 0 < y < 1$$

Find the conditional distribution of X given y.

b) Let the joint density of two random variables X and y be given by

$$f(x, y) = \frac{-3x^2 \log(y)}{2(1 + \log(2) - 2\log(4))}, \quad 0 \leq x \leq 1, \qquad 2 \leq y \leq 4$$

Are X and Y independent random variables?

Question 6

a) Let $X \sim U(-1,1)$ and let $Y = X^2$. Find Cov(X,Y).

b) Let the joint *pdf* of X and Y be

$$f(x,y) = 10, \quad for\ x < 1; x < y < x + 0.1$$

Find the correlation coefficient between X and Y

Question 7

Suppose the random variable X and Y are independent and are Gamma distributed with parameters (α, λ) and (β, λ) respectively

i.e. $f_x(x) = \frac{\lambda^\alpha x^{\alpha-1} \exp(-\lambda x)}{\Gamma(\alpha)}, \quad x \geq 0,\ \alpha, \lambda > 0$

with a similar expression for $f_Y(y)$.

By calculating the joint density function of (X+Y) and X/Y, show that these random variables are independent and that X + Y has the Gamma $(\alpha + \beta, \lambda)$ distribution. Find the probability density function of X/Y. Why are X = y and Y/X independent? What is the probability density function of Y/X?

(20 Marks)