UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2016

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 4+4+6+1+2+3]

(a) Let $g(x)$ be defined as

$$g(x) = \begin{cases} \frac{1}{3}x, & 0 \leq x < 3; \\ -\frac{1}{3}x + 2, & 3 \leq x \leq 6; \\ 0 & \text{otherwise.} \end{cases}$$

(i) If $f(x) = kg(x)$ is a probability density function, find the value of k.

(ii) Let X be a random variable with probability density function $f(x)$. Find $E(X)$. **Give enough reasons why the answer is as you have stated.**

(iii) Let Y be independent of and identically distributed as X. Find $\mathbb{P}(X + Y \leq 2)$.

(b) Let $\{X_n\}(n \geq 0)$ represent a branching process, where X_n denotes the population size in the n^{th} generation. The initial population size is 1, that is $X_0 = 1$, and in each generation the number of offspring produced by each individual that survive to the next generation has the binomial distribution with parameters 2 and $\frac{1}{2}$, so that the offspring distribution $\{p_n\}$ is given by $p_1 = \frac{1}{2}$, and $p_2 = \frac{1}{4}$. The numbers of surviving offspring produced by different individuals are statistically independent of each other.

(i) Write down an expression for the probability generating function $G(s)$ of the offspring distribution.

(ii) Find the probability of ultimate extinction $e = \lim_{n\to\infty} e_n$.

(iii) Find the $G_{X_3}(s)$.

Question 2 [20 marks, 4+3+3+3+7]

In the Ultrastratos civilisation on planet Anachronista, the population is divided into four strata which, in order of status, are labelled Alpha, Beta, Gamma and Delta. By the traditions of the civilisation, no child can have a status more than one different from its parents. As examples, in each generation 20% of the children of Alphas grow up to be Betas, the rest remaining Alphas, while of the Beta offspring 50% remain Betas while 10% become Alphas and the rest Gammas.

A Markov chain describes the status of members of the population at successive generations. Its transition matrix is given by $\mathbb{P}$, defined below, in which some entries labelled $*$ have been omitted.

$$\mathbb{P} = \begin{array}{r} Alpha \\ Beta \\ Gamma \\ Delta \end{array} \begin{pmatrix} * & 0.2 & * & 0 \\ 0.1 & 0.5 & * & * \\ * & * & 0.5 & 0.4 \\ * & 0 & * & 0.8 \end{pmatrix}$$

(a) Use the information above to fill in the missing elements of $\mathbb{P}$ and calculate the two-step-ahead transition matrix.

(b) What is the probability that the child of a Beta becomes a Gamma? Show that the grandchild of a Gamma is twice as likely to become a Delta as the grandchild of a Delta is to become a Gamma.

(c) Calculate the probability that, after four generations, a descendant of an Alpha is a Delta.

(d) Explain how you can tell that the chain consists of a single irreducible class. Are any of the states transient? Give your reason.

(e) Find the stationary distribution of the chain. If the population initially has no Deltas, what will be the proportion of Deltas after a large number of generations?

Question 3 [20 marks, 8+6+6]

(a) In a game, two independent fair dice are thrown. We repeat the throw of the two dice until we get a sum of 8, when the game stops. Two games are played in a row, and the total number of throws of the two dice is recorded in the random variable Y. Find $\mathbb{P}(Y \leq 3)$.

(b) For some $\theta > 0$, the continuous random variable X has the probability density function $f(x) = \theta e^{-\theta x} \quad (x > 0)$, that is X has an exponential distribution with expected value $\frac{1}{\theta}$ and variance $\frac{1}{\theta^2}$. Let $Y = \sqrt{X}$.

(i) Prove that Y has the Weibull distribution, with probability density function

$$g(y) = 2\theta \exp(-\theta y^2) \quad (y > 0).$$

(ii) Hence obtain exact expressions for $E(Y)$ and $\mathrm{Var}(Y)$.

Question 4 [20 marks, 8+6+6]

(a) The pair $\{X, Y\}$ has a joint density

$$f_{X,Y}(x, y) = 10y^2 x, \quad 0 \leq x \leq y \leq 1.$$

Determine the regression curve of Y on X.

(b) The number of customers in a major appliance store is equally likely to be 1, 2, or 3. Each customer buys 0, 1, or 2 items, with probabilities 0.5, 0.4, and 0.1 respectively. Customers buy independently, regardless of the number of customers. What is the probability of three or more items sold?

(c) Random variable X has moment-generating function

$$M_X(\theta) = \frac{1}{1 - 2\theta} e^{9\theta^2/2 + 4\theta}.$$

Determine $E(X)$ and $\mathrm{Var}(X)$.

Question 5 [20 marks, 6+5+9]

Suppose you apply for a passport. If you do not receive further notice from the related government department (with probability $1/2$), the waiting time T_1 until you receive your new passport is exponentially distributed with parameter λ, that is $T_1 \sim \exp(\lambda)$, with probability density function

$$f_{T_1}(t) = \lambda \exp(-\lambda t), \quad t > 0.$$

However, there is a probability of $1/3$ that there is a minor delay in processing your application, and the waiting time T_2 until you receive your new passport will then be $T_2 \sim \exp(\lambda/2)$.

There is also a probability of $1/6$ that there is a serious delay. The waiting time T_3 until you receive your new passport will then be $T_3 \sim \exp(\lambda/4)$.

All time units are in months.

(a) Let T be the overall waiting time until you receive the new passport. Find $\mathbb{P}(T < 4)$ in terms of λ.

(b) You have waited for 4 months and still have not received your passport. What is the probability (in terms of λ) that there are actually delays (minor or serious) in your application?

(c) Suppose $\lambda = 1$. Find the mean and variance of T. (You can use the mean and variance of an exponential random variable without proof, as long as you state them clearly. Hint: For the variance of T, find $E(T^2)$ first.)

Question 6 [20 marks, 4+3+5+8]

(a) The joint probability density for the random variable X and Y is given by

$$f_{X,Y}(x,y) = k\left(\frac{y}{x+y}\right)^{1/2}, \qquad 0 < x < y < 1.$$

Show that $k = 1 + \sqrt{2}$.

(b) In a fast food restaurant, the manager is interested in the total number of chicken wings sold in a day. The number of chicken wings, X_i, ordered by customer i for $i = 1, \cdots, N$ is independent of each other, and has the following distribution:

$$X_i = \begin{cases} 0, & \text{with probability } p; \\ Y_i, & \text{with probability } q = 1 - p, \end{cases}$$

where each Y_i's is independent of all other random variables, and has probability mass function

$$p_{Y_i}(y) = \frac{\mu^y e^{-\mu}}{Cy!}, \quad 1, 2, \cdots,$$

for some constant $C > 0$.

(i) Find the constant C.

(ii) Write down T, the total number of chicken wings sold in a day, in terms of X_i and N, where N is the total number of customers in a day. Show that the moment generating function of X_i is

$$M_{X_i}(t) = p + \frac{qe^{-\mu}}{1 - e^{\mu}}(\exp(\mu e^t) - 1), \quad t \in \mathbb{R}.$$

(iii) If N is independent of all other random variables and is Poisson distributed with mean n, by conditioning on N first, derive the moment generating function of T. Hence, or otherwise, find the mean of T.